Predictive Sets

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Can you guess?

Suppose I give you a sequence: 1,

Can you guess?

Suppose I give you a sequence: 1,1,

Can you guess?

Suppose I give you a sequence: 1,1, 1,

Can you guess?

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Can you guess?

Suppose I give you a sequence: 1,1, 1, 1, 1, 1,

What comes next?

Can you guess?

Suppose I give you a sequence: 1,1, 1, 1, 1, 1,

What comes next?

It is probably going to be 1.

Can you guess?

What about this one: 1,

Can you guess?

What about this one: 1,2,

Can you guess?

What about this one: 1,2, 3,

Can you guess?

What about this one: 1,2, 3, 1,

Can you guess?

What about this one: 1,2, 3, 1, 2,

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

It is probably going to be 1 again.

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

It is probably going to be 1 again.

But it could very well have been part of

$$\ldots, 4, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 1, 2, 3, 4, \ldots$$

in which case it should have been 4.

Be careful with your guesses.

We know that without enough information about how the sequence comes about there is not much point in guessing.

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We know that without enough information about how the sequence comes about there is not much point in guessing.

But what if instead I give you the entire past of the sequence and tell you before hand that the sequence is periodic. Then we can always predict precisely.

Do we need to know the entire past to make this prediction?

Predicting periodic sequences

Clearly, it would be enough to know the sequence along the even integers because the restriction of periodic sequence to the even integers is still periodic.

$$\dots, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \dots$$

$$?, \times, 1, \times, 3, \times, 1, \times, 3, \times, 1, \times, 3, \times, \dots$$

Predicting periodic sequences

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Clearly it is not enough to know the sequence along the odd integers.

$$?, 1, \times, 3, \times, 1, \times, 3, \times, 1, \times, 3, \times, \ldots$$

We do not know after all which periodic sequences runs along the odds.

Can we cut down further?

Predicting periodic sequences

A set $Q \subset \mathbb{N}$ is called a PER-set if $Q = \{nk \ : \ k \in \mathbb{N}\}$ for some $n \in \mathbb{N}$.

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Even integers are PER* but odd integers are not PER*.

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Suppose $x_i; i \in \mathbb{Z}$ is a periodic sequence with period p.

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Now suppose that x_i is constant for $i \in P \cap \{nk \ : \ k \in \mathbb{N}\}$ for some $n \in \mathbb{N}$. But P is PER*. Hence it also contains a multiple of np.

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Now suppose that x_i is constant for $i \in P \cap \{nk \;:\; k \in \mathbb{N}\}$ for some $n \in \mathbb{N}$. But P is PER*. Hence it also contains a multiple of np.

Hence we can decide what x_0 is, given $x_i; i \in P$.

In other words, a set can predict all periodic sequences if and only if it is PER*.

Discrete spectrum

This idea can be generalised to processes with discrete spectrum.

Stationary Processes

A stationary process is a sequence of random variables $\ldots, X_{-2}, X_{-1}, X_0, \ldots$ such that the distribution of

$$X_0, X_1, \ldots, X_n$$

is the same as that of

$$X_k, X_{k+1}, \ldots, X_{k+n}.$$

Our processes will always be finite valued unless otherwise mentioned.

What is discrete spectrum?

A stationary process with the shift map is said to have discrete spectrum if it is isomorphic to the rotation of a compact abelian group (with the Haar measure).

Discrete spectrum

A process is said to have discrete spectrum if it is isomorphic to the rotation of a compact abelian group (with the Haar measure).

For instance if $x_i; i \in \mathbb{Z}$ is a periodic point with periodic p then its orbit (under the shift map), is isomorphic to the rotation of $\mathbb{Z}/p\mathbb{Z}$ by 1.

Circle rotations give another example.

Rotations

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the circle. Given $x, \alpha \in \mathbb{T}$ we consider the rotation $x, x+\alpha, x+2\alpha, \ldots$.

We split the circle into two parts $[0, 1/2)$ and $[1/2, 1)$. If the point falls on the first part we record a 0 and if it falls on the second half we record a 1.

Thus starting with a point x we get a sequence in 0 and 1.

Rotations

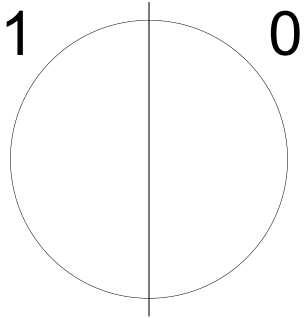

Figure : Recording a circle rotation by 0 and 1:

Rotations

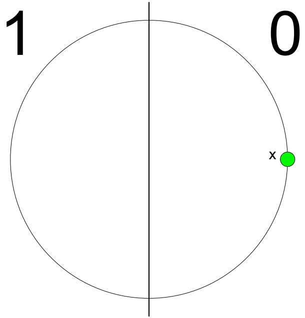

Figure : Recording a circle rotation by 0 and 1: 0

Rotations

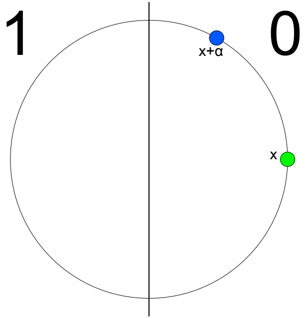

Figure : Recording a circle rotation by 0 and 1: 0, 0

Rotations

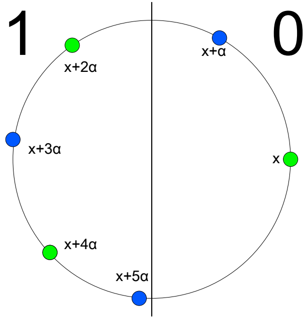

Figure : Recording a circle rotation by 0 and 1: 0, 0, 1, 1, 1, 1

DIS* sets

A set Q is called DIS-set if it there is a compact abelian group K, $\alpha \in K$ and open set $0 \in U \subset K$ such that

$$Q = \{n \in \mathbb{N} \ : \ n\alpha \in U\}.$$

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Every PER-set (the set of multiples of an integer) is a DIS-set for the map $r : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by

$$r(x) := x + 1.$$

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$$r(x) := x + 1.$$

A set P is called DIS* it it intersects every DIS-set.

Predicting processes with discrete spectrum

Theorem

Let $P \subset \mathbb{N}$. We can predict X_0 given $X_i; i \in P$ for all processes $X_i; i \in \mathbb{Z}$ with discrete spectrum if and only if P is DIS.*

Let us now discuss more general processes.

Deterministic Processes: Examples

A stationary process $X_{\mathbb{Z}}$ is called deterministic if there is a measurable function $\Phi : A^{-\mathbb{N}} \to A$ such that

$$\Phi(X_{-\mathbb{N}}) = X_0$$

with probability one.

If a process is periodic almost surely, then it is deterministic.

In general, all processes with discrete spectrum are deterministic.

Deterministic Processes: Non-examples

A stationary process $X_{\mathbb{Z}}$ is called deterministic if there is a measurable function $\Phi : A^{-\mathbb{N}} \to A$ such that

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with probability one.

The process given by coin tosses is not deterministic.

Predictive Sets

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

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Predictive Sets: Examples

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

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Thus by definition $\mathbb{N}$ is a predictive set.

Predictive Sets: Examples

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

$$\Phi(X_{-P}) = X_0.$$

Suppose $\mathbb{N} + k \subset P$ then P is predictive. Because then by stationarity, X_{-P} can predict X_{-k} and hence X_{-k+1} and subsequently X_0.

Predictive Sets: Non-examples

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

$$\Phi(X_{-P}) = X_0.$$

The set P of odd numbers cannot even predict periodic sequences; it is not predictive.

In fact, there are weak-mixing zero-entropy processes $X_{\mathbb{Z}}$ where X_0 is independent of X_P for the set of odd numbers P.

The proof goes via Riesz products and Gaussian processes.

Even numbers

Are the even numbers predictive?

Even numbers

Are the even numbers predictive?

For this we need to introduce some entropy theory.

Entropy of random variables

Shannon revolutionised information theory in 1948 by bringing in a host of new ideas and technology.

At the centre of this revolution was entropy.

For a random variable X taking values in the set A, the entropy of X is given by

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a))$$

assuming $0 \log 0 = 0$.

Shannon Entropy

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)).$$

Clearly $\mathbb{P}(X = a) \log(\mathbb{P}(X = a)) \geq 0$. It is equal to 0 if and only if $\mathbb{P}(X = a) = 1$ for some $a \in A$.

Thus $H(X) = 0$ if and only if it is completely determined.

Shannon Entropy

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)).$$

On the other hand $\theta \to -\log \theta$ is a convex function. By Jensen's inequality we have

$$\begin{aligned} H(X) &:= -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)) \\ &\leq \log(\sum_{a \in A} \frac{\mathbb{P}(X = a)}{\mathbb{P}(X = a)}) \\ &= \log(|A|) \end{aligned}$$

with equality if and only if $\mathbb{P}(X = a) = \frac{1}{|A|}$ for all $a \in A$.

Shannon Entropy

Thus $H(X) = 0$ if and only if X is deterministic and

$H(X) \leq \log(|A|)$ with equality if and only if X is uniformly distributed.

Shannon Entropy

Further $H(X|Y) := H(X, Y) - H(Y)$ and similarly one can prove

$H(X|Y) = 0$ if and only if X is a function of Y.

Kolmogorov Sinai Entropy

For a process $X_{\mathbb{Z}}$, Kolmogorov Sinai entropy is defined by the limit

$$
\begin{aligned}
h(X_{\mathbb{Z}}) &:= \lim_{n\to\infty} \frac{1}{n} H(X_0, X_1, \ldots, X_{n-1}) \\
&= \lim_{n\to\infty} \frac{1}{n} (H(X_0) + H(X_1|X_0) + \ldots, H(X_{n-1}|X_{n-2}, \ldots, X_0))) \\
&= \lim_{n\to\infty} \frac{1}{n} (H(X_0) + H(X_0|X_{-1}) + \ldots, H(X_0|X_{-1}, \ldots, X_{-n+1})) \\
&= H(X_0|X_{-1}, X_{-2}, \ldots).
\end{aligned}
$$

Thus $h(X_{\mathbb{Z}}) = 0$ if and only if $H(X_0|X_{-1}, X_{-2}, \ldots) = 0$ if and only if X_0 is a function of $X_{-\mathbb{N}}$.

Back to the evens

$X_{\mathbb{Z}}$ is deterministic if and only if $h(X_{\mathbb{Z}}) = 0$.
It is easy to see that

$$\begin{aligned} h(X_{\mathbb{Z}}) &:= \lim_{n\to\infty} \frac{1}{n} H(X_0, X_1, \ldots, X_{n-1}) \\ &\geq \lim_{n\to\infty} \frac{1}{n} H(X_0, X_k, X_{2k} \ldots,) \\ &\geq \frac{1}{k} h(X_{k\mathbb{Z}}). \end{aligned}$$

Thus if $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

Back to the evens

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Thus if $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

Caution: This is not true unless the state space is finite.

Back to the evens

If $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

We know that $\mathbb{N}$ is predictive. Thus we have that $k\mathbb{N}$ is also a predictive set.

Some sufficient conditions.

Return-time sets are predictive

Let (X, μ, T) be a probability preserving transformation (ppt). Given a set $U \subset X$ of positive measure, we denote by

$$N(U, U) := \{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}.$$

A set $A \subset \mathbb{N}$ is called a return-time set if $A = N(U, U)$ for some ppt.

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Return-time sets are predictive

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Note that $k\mathbb{N}$ is a return-time set for the transformation $T : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ given by $T(i) = i + 1$.

Thus we have generalised our former observation that $k\mathbb{N}$ is predictive.

Return-time sets are predictive

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

This theorem can be formally strengthened for return-time sets coming from zero-entropy ppt. If (X, μ, T) is a zero entropy ppt, $U \subset X$ with $\mu(U) > 0$ and P is a predictive set then $P \cap N(U, U)$ is also a predictive set.

Question

Does every return-time set contain a return-time set of a zero-entropy process?

The intersection of a return-time set of a zero entropy process and a predictive set is predictive

If (X, μ, T) is a zero entropy ppt, $U \subset X$ with $\mu(U) > 0$ and P is a predictive set then $P \cap N(U, U)$ is also a predictive set.

It is easy to see that if $\alpha \in \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$ then the set

$$\{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}$$

contains a return-time set for $U = (-\epsilon/2, \epsilon/2)$.

Thus if P is predictive then

$$P \cap \{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}$$

is also predictive.

Results: SIP^*

Given a sequence $s_1, s_2, \ldots$ we write

$$SIP(s_1, s_2, \ldots) := \{\sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\}\} \cap \mathbb{N}.$$

A set $P \subset \mathbb{N}$ is called $SIP^\star$ if it intersects every SIP set.

Clearly one of the numbers of the type $\sum_{i=1}^{\infty} \epsilon_i s_i$ is even for all sequences $s_1, s_2, \ldots$ (either s_1, s_2 or $s_1 + s_2$ has to be even).

Thus the even numbers are $SIP^\star$ and similarly $k\mathbb{N}$ for all $k \in \mathbb{N}$.

The odd numbers are not $SIP^\star$. They do not intersect the SIP generated by even numbers.

Results: *SIP**

Theorem

Predictive sets are SIP*.

Given an SIP Q, we construct a weak-mixing process $X_{\mathbb{Z}}$ for which X_0 is independent of $X_{\mathbb{N} \setminus Q}$.

An easy consequence is that predictive sets P have bounded gaps meaning $\mathbb{N} \setminus P$ cannot contain intervals of unbounded length.

Let us see why.

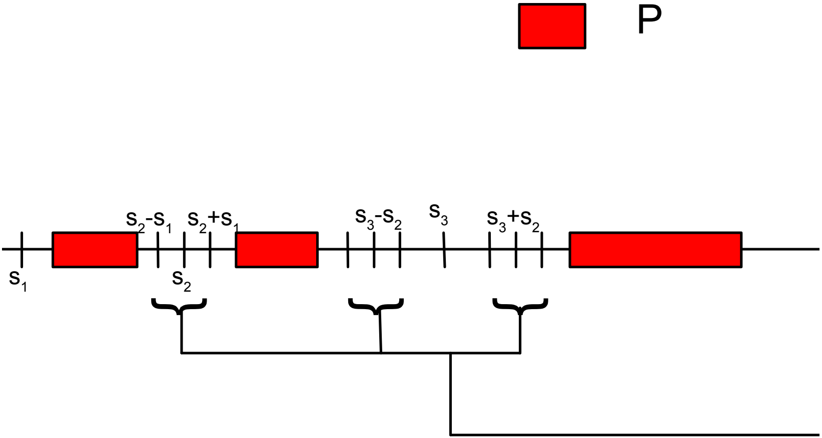
$$SIP(s_1, s_2, \ldots) := \{\sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\}\} \cap \mathbb{N}.$$

Predictive sets intersect every SIP set.

Suppose P is a predictive set such that $\mathbb{N} \setminus P$ contains intervals of unbounded length.

Then we can fit an SIP set in $\mathbb{N} \setminus P$.

Fitting SIP sets in $\mathbb{N} \setminus P$ if P has does not have bounded gaps



Sufficient conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Necessary conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Predictive sets are SIP*.

The following question arises naturally.

Question

Are sufficient conditions necessary and necessary conditions sufficient?

Let us give some partial answers.

Are all $SIP^\star$ sets predictive?

If P is a predictive set, $\epsilon > 0$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ then

$$\{n \in \mathbb{N} \ : \ n\alpha \in (-\epsilon, \epsilon)\} \cap P$$

is predictive.

Question

Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive?

An uncertain theorem

Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive?

If the answer is yes then we have two predictive sets

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\} \text{ and } \{n \in \mathbb{N} \ : \ -n\alpha \in (0, \epsilon)\}$$

which do not intersect.

Theorem (Akin and Glasner, 2016)

The set $\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$ is $SIP^{\star}$.

Thus if the answer is no then we have a $SIP^{\star}$ set which is not predictive.

So we don't really know if all $SIP^\star$ sets are predictive.

There are predictive sets which do not contain return-time sets.

Consider the set
$$Q = \{n^2 \ : \ n \in \mathbb{N}\}.$$

For all $i, k \in \mathbb{N}$ we have that if
$$n^2 = -i + 3i^2k = i(-1 + 3ik)$$
then since i and $-1 + 3ik$ are prime to each other, they are perfect squares.

But this is impossible because $-1 + 3ik \equiv -1 \pmod 3$. Thus $\mathbb{N} \setminus Q$ contains $-i + 3i^2k; k \in \mathbb{N}$.

There are predictive sets which do not contain return-time sets.

Hence we have that

$$H(X_{-i} \mid X_{\mathbb{N} \setminus Q}) = 0$$

for all $i \in \mathbb{N}$.

But then for all $i \in \mathbb{Z}$

$$H(X_i \mid X_{\mathbb{N} \setminus Q}) = H(X_i \mid X_{(-\mathbb{N}) \cup (\mathbb{N} \setminus Q)}) = 0.$$

But all return-time sets must intersect the set $\{n^2 \ : \ n \in \mathbb{N}\}$ (Sarkozy, Furstenberg). Thus there are predictive sets which are not return-time sets.

Predictive sets

Question

Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \mid k\in\mathbb{N}\}}) = 0.$$

We do not know this even in the case $n_k = k^3$. We will come back to this later if time permits.

Proofs.

Return-time sets are predictive

Let (X, μ, T) be a ppt and $U \subset X$ have positive measure. We will prove that

$$\{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}$$

is predictive.

Using the ergodic theorem, it can be proved that these return-time sets contain the difference set of a positive density set.

It is sufficient to prove that the difference set of a positive density set is predictive.

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n \to \infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \ldots\}$ have density

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and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \ldots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \ldots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

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But

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \ldots, X_{q_n}) \quad = \quad \frac{1}{n} H(X_0 \mid X_{q_2 - q_1}, X_{q_3 - q_1}, \ldots, X_{q_n - q_1})$$

Return-time sets are predictive

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and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

But

$$\begin{aligned} \frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n} H(X_0 \mid X_{q_2 - q_1}, X_{q_3 - q_1}, \dots, X_{q_n - q_1}) \\ &+ \frac{1}{n} H(X_0 \mid X_{q_3 - q_2}, X_{q_4 - q_2}, \dots, X_{q_n - q_2}) + \dots \end{aligned}$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n}H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \le \frac{q_n}{n}\frac{1}{q_n}H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha}h(X_{\mathbb{Z}}) = 0.$$

But

$$\begin{aligned} \frac{1}{n}H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n}H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1}) \\ &+ \frac{1}{n}H(X_0 \mid X_{q_3-q_2}, X_{q_4-q_2}, \dots, X_{q_n-q_2}) + \dots \\ &+ \frac{1}{n}H(X_0 \mid X_{q_n-q_{n-1}}) \end{aligned}$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

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&+ \frac{1}{n} H(X_0 \mid X_{q_n-q_{n-1}}) \\
&\geq H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}})
\end{aligned}$$

Return-time sets are predictive

Thus if Q has positive density then

$$H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}}) = 0$$

and $(Q-Q)\cap\mathbb{N}$ is a predictive set. We showed earlier that every return-time set contains such a set.

Thus return-time sets are predictive.

Predictive sets are $SIP^\star$

In course of the proof we show that for all $SIP(S)$ there exists a weak mixing zero entropy Gaussian process $X_{\mathbb{Z}}$ such that

$$X_0 \text{ is independent of } X_i \text{ for } i \in \mathbb{N} \setminus SIP(S).$$

This shows that $\mathbb{N} \setminus SIP(S)$ is not predictive.

Thus there exists a weak-mixing process in which X_0 can be predicted by $X_{\mathbb{N}}$ but is independent of $X_{2\mathbb{N}+1}$.

Predictive sets are $SIP^\star$: Processes and Spectral measures

From here on we will assume that X_0 is complex-valued, has zero mean and finite variance.

Given any process $X_{\mathbb{Z}}$, the sequence $\mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N}$ is a positive definite sequence.

By Herglotz's theorem, there exists a probability measure μ on $\mathbb{R}/\mathbb{Z}$ such that the Fourier coefficients

$$\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

On the other hand, given any probability measure μ on $\mathbb{R}/\mathbb{Z}$ there exists a Gaussian process $X_{\mathbb{Z}}$ such that

$$\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

Predictive sets are $SIP^{\star}$: Processes and Spectral measures

$X_{\mathbb{Z}}$

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Predictive sets are $SIP^\star$: Processes and Spectral measures

$$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu \text{ on } \mathbb{R}/\mathbb{Z} \text{ such that } \hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

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μ on $\mathbb{R}/\mathbb{Z}$

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$$\mu \text{ on } \mathbb{R}/\mathbb{Z} \longrightarrow \text{ Gaussian process } X_{\mathbb{Z}} \text{ for which } \hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

Predictive sets are $SIP^\star$: Processes and Spectral measures

$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu$ on $\mathbb{R}/\mathbb{Z}$ such that $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

μ on $\mathbb{R}/\mathbb{Z} \longrightarrow$ Gaussian process $X_{\mathbb{Z}}$ for which $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

If μ is singular then $X_{\mathbb{Z}}$ has zero entropy (Newton and Parry).

For Gaussian processes X_0 and X_n are independent if and only if $\hat{\mu}(n) = 0$.

A Gaussian process $X_{\mathbb{Z}}$ is weak-mixing if and only if μ is continuous.

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Predictive sets are $SIP^\star$: Riesz products

Fix a sequence $s_1, s_2, \ldots \subset \mathbb{N}$ such that $s_{i+1} > 3s_i$.

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The Riesz product is the function $f_r : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ given by

$$\begin{aligned} f_r(x) &:= \prod_{k \le r} (1 + \cos(2\pi s_k x)) \\ &= \prod_{k \le r} \left(1 + \frac{\exp(2\pi i s_k x) + \exp(-2\pi i s_k x)}{2}\right). \end{aligned}$$

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As r tends to infinity the limit of $f_r \mu_{Leb}$ is a singular continuous measure μ such that $\hat{\mu}(n) = 0$ for all

$$n \notin SIP(s_1, s_2, \ldots) := \left\{ \sum_{t \in \mathbb{N}} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\} \right\}.$$

Thus $X_{\mathbb{Z}}$ has zero entropy, is weak mixing and $\mathbb{E}(X_0 \overline{X_n}) = 0$ for all

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Predictive sets are $SIP^\star$

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$$n \notin SIP(s_1, s_2, \ldots).$$

If P is predictive then

$$P \cap SIP(s_1, s_2, \ldots) \neq \emptyset.$$

One can use this to prove that predictive sets are SIP^*.

Linear Predictivity

In fact if μ is singular by a theorem of Verblunsky we get the following result:

Theorem

If $X_{\mathbb{Z}}$ is a complex-valued L^2 process for which the spectral measure μ is singular and P is predictive then X_0 is in the closed linear span of $X_i; i \in P$.

I wasn't aware of this even for processes arising from circle rotations and $P = \mathbb{N}$.

On the other hand if the spectral measure has a Lebesgue component but $X_{\mathbb{Z}}$ has zero entropy then we can predict but not linearly predict the process.

Riesz Sets

Using this machinery we can conclude the following result.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a set such that $P + i$ is predictive for all $i \in \mathbb{N}$ then for all singular measures μ on $\mathbb{R}/\mathbb{Z}$ there exists $p \in P$ such that the Fourier coefficient

$$\hat{\mu}(p) \neq 0.$$

In other words any measure μ on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on $\mathbb{Z} \setminus P$ must have an absolutely continuous component.

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This is very close to Riesz sets as defined by Yves Meyer in 1968: A set $Q \subset \mathbb{Z}$ is called a Riesz set if all measures on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on Q are absolutely continuous.

Riesz Sets

A set $P \subset \mathbb{N}$ is called totally predictive if $P + i$ is predictive for all $i \in \mathbb{N}$.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a totally predictive set which is open in the Bohr topology, then $\mathbb{Z} \setminus P$ is a Riesz set.

Question

If $P \subset \mathbb{N}$ is totally predictive then is $\mathbb{Z} \setminus P$ a Riesz set? If $Q \subset \mathbb{N}$ is a set such that $Q \cup (-\mathbb{N})$ is Riesz then is $\mathbb{N} \setminus Q$ a totally predictive set?

A titillating question

Let $n_{\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_{i+1} - n_i$ is also an increasing sequence. We had asked whether $\mathbb{N} \setminus n_{\mathbb{N}}$ is totally predictive.

It is unknown even for $n_i = i^3$ whether $(-\mathbb{N}) \cup n_{\mathbb{N}}$ is a Riesz set. Wallen (1970) proved that if μ is a measure whose Fourier coefficients are supported on $(-\mathbb{N}) \cup n_{\mathbb{N}}$ then $\mu \star \mu$ is absolutely continuous.

Following an idea by Lindenstrauss, a simple application of Fermat's last theorem and Cauchy Schwarz gives us the following partial result.

Theorem (Chandgotia, Weiss)

If μ is a probability measure whose Fourier coefficients are supported on $\{\pm i^K : i \in \mathbb{N}\} \cup \{0\}$ for some $k \geq 2$ then μ is not singular.

Summary

Return-time sets are predictive.

The converse is not true.

Predictive sets are $SIP^{\star}$.

Predictive sets have bounded gaps.

If you were bored...

1. Is the intersection of two predictive sets also a predictive set?
2. Are all $SIP^\star$ sets predictive?
3. Is $\{n \ : \ n\alpha \in (0, \epsilon)\}$ a predictive set for irrational α?
4. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \ \mid \ k\in\mathbb{N}\}}) = 0.$$

5. What is the relationship between Riesz sets and totally predictive sets?
6. Explore linear prediction.