The realization problem of homeomorphisms in $\mathbb{R}^3$ for toroidal sets

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Statement of the problem

Problem

Given a compactum $K \subset \mathbb{R}^3$ determine whether there exists a homeomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that K is an attractor for f.

Some remarks

We are also interested in the analogous realization problem for flows:

Problem

Given a compactum $K \subset \mathbb{R}^3$ determine whether there exists a flow $\varphi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3$ such that K is an attractor for φ.

Some remarks

The realization problem posed is different in nature from the abstract realization problem namely:

Problem

Given a compact metric space K determine whether there exists an embedding $e : K \longrightarrow \mathbb{R}^n$ for some n such that $e(K)$ is an attractor for a homeomorphism (or flow) on $\mathbb{R}^n$.

Some remarks

While the abstract problem depends only on the topology of K, the realization problem we are dealing with depends not only on the topology of K but on the way in which K sits on $\mathbb{R}^3$.

Some remarks

Notice that our results hold for homeomorphisms (or flows) globally defined on $\mathbb{R}^3$ and that the analogous results in manifolds other than $\mathbb{R}^3$ or for local homeomorphisms of $\mathbb{R}^3$ may be false.

Attractors

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a homeomorphism. A compactum $K \subset \mathbb{R}^3$ is said to be an *attractor* for f if it satisfies the following properties:

1. K is invariant,
2. K is stable in the sense of Lyapunov,
3. K possessess a neighborhood U whose points are attracted by K.

Attractors

If K is an attractor, the maximal neighborhood of K whose points are attracted by K is an open set called *basin of attraction* of K and denoted by $\mathcal{A}(K)$.

A useful property of attractors

We shall make use of the following property :

Property

Let $P \subset \mathcal{A}(K)$ be a compactum. Then, for every neigborhood V of K there exists $n_0 \geq 0$ such that

$$f^n(P) \subset V \quad \text{for every } n \geq n_0.$$

Special bases of neighborhoods

Let $V \subset \mathcal{A}(K)$ a compact neighborhood of K. Then there exists $n_0 \geq 0$ such that $f^{n_0}(V) \subset \mathring{V}$. Let $g = f^{n_0}$ and consider the family $\{V_n\}_{n \geq 0}$ given by

$$V_n = g^n(V).$$

Special bases of neighborhoods

The family $\{V_n\}_{n\geq 0}$ has the following properties:

Properties

1. $V_{n+1} \subset \mathring{V}_n$ *for every* $n \geq 0$.
2. $K = \bigcap_{n\geq 0} V_n$.
3. V_i *and* V_j *are ambient homeomorphic (hence homeomorphic) for every* $i, j \geq 0$.
4. *The pairs* (V_i, V_{i+1}) *and* (V_j, V_{j+1}) *are homeomorphic for every* $i, j \geq 0$.

Special bases of neighborhoods

In the case of flows we can do even better. The existence of Lyapunov functions allows us to get a family $\{V_n\}_{n\geq 0}$ with the previous properties and the following additional property:

Property

$\overline{V_i \setminus V_{i+1}} \approx \mathrm{Fr}\, V_i \times [0,1]$ *for every* $i \geq 0$ *(concentricity).*

Moreover, the existence of such a family qualifies K to be an attractor of a flow (hence of a homeomorphism).

How to construct compacta that cannot be attractors

The previous construction allows us to identify three kinds of compacta $K \subset \mathbb{R}^3$ that cannot be attractors.

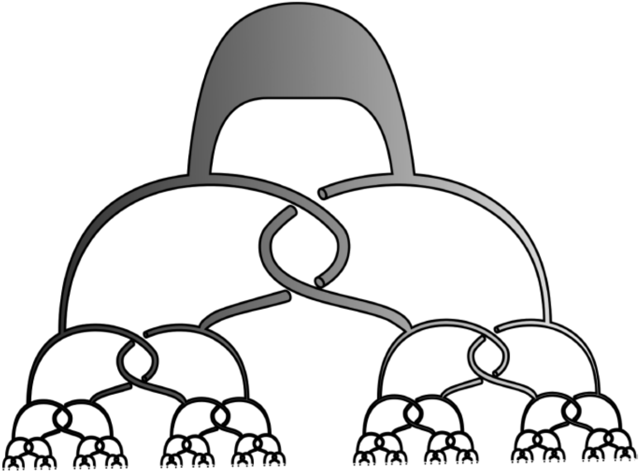
A) Compacta that do not have a basis of neighborhoods comprised of homeomorphic compacta.

B) Compacta that do not have a basis of neighborhoods comprised of ambient homeomorphic compacta.

C) Compacta whose nested neighborhood bases do not satisfy that consecutive elements lie in the same way.

Example

A suggestive example of a compactum in the A) case is the Alexander Horned Sphere.



Cellular sets

From now on we are going to assume that our compacta have neighborhood bases comprised of homeomorphic elements. The easiest case would be to consider that the elements of the bases are 3-cells. Compacta that have this property are the so-called *cellular sets*.

Cellular sets

It is known that cellullar sets can be realized as (global) attractor of flows on $\mathbb{R}^3$. This is not very surprising since due to the PL Schöenflies Theorem there is only one way to place a polyhedral cell in $\mathbb{R}^3$.

Toroidal sets

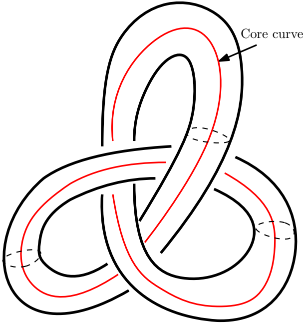
The next natural case is when K has a neighborhood basis comprised of solid tori.

Definition

We say that a compactum $K \subset \mathbb{R}^3$ is *toroidal* if it is not cellular and possesses a basis of neighborhoods comprised of solid tori.

Knotted solid tori

A solid torus $T \subset \mathbb{R}^3$ is said to be *unknotted* if its core curve is unknotted. Otherwise we say that T is a *knotted* solid torus.



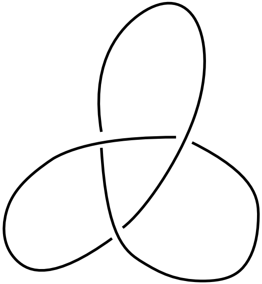
Knotted toroidal sets

Definition

We say that a toroidal set $K \subset \mathbb{R}^3$ is *unknotted* if it possesses a basis of neighborhoods comprised of unknotted solid tori. Otherwise we say that K is *knotted*.

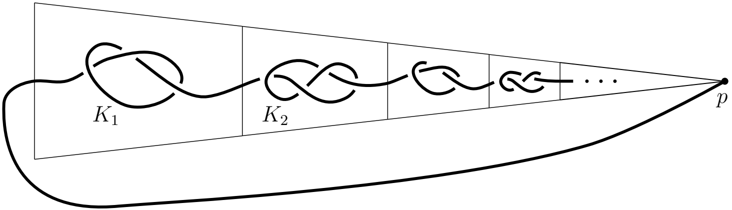
Tame knots

Any tame non-trivial knot K is a knotted toroidal set since every regular neighborhood of K in the PL sense is a solid torus.



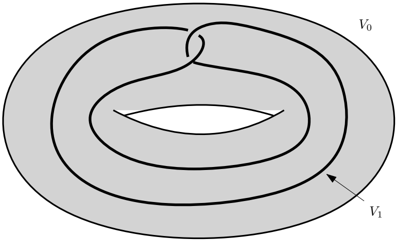
Infinite connected sums of tame knots

Any knot constructed as an infinite connected sum of non-trivial knots with one wild point is an example of knotted toroidal set.



The Whitehead continuum

Consider the standard solid torus $V_0 \subset \mathbb{R}^3$ and consider a second solid torus $V_1 \subset \mathrm{int}\ V_0$ as depicted in the picture.



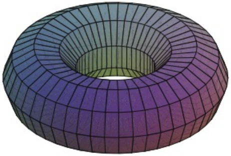
Consider another solid torus $V_2 \subset \mathrm{int}\ V_1$ placed in the same pattern and continue the construction *ad infinitum* obtaining a nested family of solid tori $\{V_n\}_{n \geq 0}$. Then

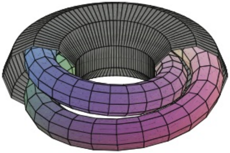
$$K = \bigcap_{n \geq 0} V_n$$

is an unknotted toroidal set known as the *Whitehead continuum*.

Generalized solenoids

Consider a nested family of solid tori $\{T_n\}_{n\geq 0}$ such that T_{i+1} wind $n_i \geq 2$ times inside T_i monotonically and the thickness of T_i goes to 0 as i increases.



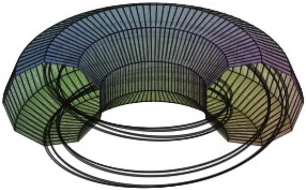


Generalized solenoids

The compactum

$$K = \bigcap_{n \geq 0} T_n.$$

is the so-called *generalized solenoid*.



In case that $n_i = n$ for i sufficiently large we say that K is an n-adic solenoid.

Special bases for toroidal attractors

If $K \subset \mathbb{R}^3$ is a toroidal set that is an attractor of some homeomorphism of $\mathbb{R}^3$ choosing a solid torus $T \subset \mathcal{A}(K)$ that is a neighborhood of K we obtain a basis of neighborhoods $\{T_n\}_{n\geq 0}$ comprised of solid tori with the following properties:

Properties

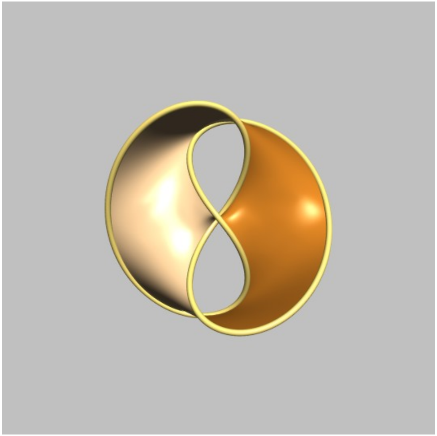
1. *$T_{n+1} \subset \mathrm{int}\ T_n$ for every $n \geq 0$.*
2. *T_i and T_j are knotted in the same way for every $i, j \geq 0$.*
3. *The patterns (T_i, T_{i+1}) and (T_j, T_{j+1}) are the same.*

Useful invariants

1. To study the degree of knotteness of our solid tori we shall use a classical invariant from knot theory known as the *genus*.
2. In order to study the patterns (T, T') we shall make use of two invariants:
   - The *winding number*: An algebraic count of the number of the revolutions that T' gives inside T.
   - The *geometric index*: Minimun number of points of intersection of a core γ of T' with any meridian of T that intersects γ transversally.

Seifert surfaces

It is well known that given a knot $L \subset \mathbb{R}^3$ there exists a compact surface with boundary $S \subset \mathbb{R}^3$ such that $\partial S = L$. A surface satisfying this condition is called *Seifert surface* spanning L.



Genus of a knot

The *genus* of a knot L is defined to be the minimum of the genera among all the Seifert surfaces spanning L. The genus is a knot invariant that characterizes the unknot, that is:

Property

L is the unknot if and only if its genus is zero.

Genus of a toroidal set

The genus of a solid torus $T \subset \mathbb{R}^3$ is defined as the genus of its core.

Definition

Let $K \subset \mathbb{R}^3$ be a toroidal set. We define the *genus* of K, $g(K)$, as the minimum g among $0, 1, 2, \ldots, \infty$ such that K has arbitrarily small neighborhoods that are polyhedral solid tori of genus$\leq g$.

The following property is a direct consequence of the fact that the genus of a knot determines the unknot.

Property

A toroidal set $K \subset \mathbb{R}^3$ is unknotted if and only if $g(K) = 0$.

Genus of a toroidal attractor

Theorem

Let $K \subset \mathbb{R}^3$ be a toroidal set that is an attractor of a homeomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Then K has finite genus.

Idea of the proof

All the tori of a basis constructed with f are knotted in the same way and, hence, have the same genus.

Winding numbers and cohomology

Let $K \subset \mathbb{R}^3$ be a toroidal set and $\{T_n\}_{n\geq 0}$ a nested basis of neigborhoods of K comprised by solid tori. We recall that the winding number w_i of T_{i+1} inside T_i is the number or revolutions that T_{i+1} gives inside T_i counted algebraically. This number coincides with

$$\mathbb{Z} \cong H^1(T_i) \xrightarrow{\cdot w_i} H^1(T_{i+1}) \cong \mathbb{Z}$$

where the previous homomorphism is induced by the inclusion $j : T_{i+1} \hookrightarrow T_i$.

Toroidal sets according to its Čech cohomology

Using the continuity property of Čech cohomology we get

$$\check{H}^1(K) = \varinjlim\{H^1(T_i) \xrightarrow{\cdot w_i} H^1(T_{i+1})\}$$

and, as a consequence we obtain three mutually exclusive posibilities:

- $\check{H}^1(K) = 0$ whenever $w_i = 0$ for infinitely many i.
- $\check{H}^1(K) = \mathbb{Z}$ if $w_i = 1$ for i sufficiently large.
- $\check{H}^1(K)$ is not finitely generated otherwise.

Those toroidal sets with $\check{H}^1 = 0$ are called *trivial toroidal sets*.

Examples

- The Whitehead continuum is a trivial toroidal set.
- Any toroidal knot K satisfies $\check{H}^1(K) = \mathbb{Z}$.
- Generalized solenoids have $\check{H}^1$ not finitely generated.

Genus of non-trivial toroidal sets

Theorem

The genus of a non-trivial toroidal set is

$$g(K) = \lim_{i \to \infty} g(T_i)$$

where $\{T_i\}$ is any neighborhood basis comprised of nested polyhedral solid tori.

A useful formula

The following formula is fundamental in the proof of the previous result:

Property

Let $T', T \subset \mathbb{R}^3$ *be solid tori with* $T' \subset \mathrm{int}(T)$ *and winding number* $w \neq 0$. *Then*

$$g(T') \geq w \cdot g(T) + g(T, T').$$

Example: Polyhedral knots

Let K be a polyhedral knot. If we choose a basis of nested regular neighborhoods $\{T_n\}_{n\geq 0}$ of K we have that for every i, $g(T_i)$ coincides with the genus of K as knot and, therefore $g(K)$ also agrees with the genus of K as a knot.

Example: Infinite connected sums of knots

Let K be a toroidal set that is an infinite connected sum of non-trivial knots. Then $g(K) = \infty$. As a consequence, infinite connected sums of non-trivial knots cannot be realized as attractors of homeomorphisms of $\mathbb{R}^3$.

Knots that cannot be attractors

Corollary

There are uncountable many inequivalent ways of embedding S^1 in $\mathbb{R}^3$ in such a way that it cannot be realized as an attractor for a homeomorphism. All these embeddings are tame except for a single point.

Example: Non-trivial toroidal sets with $\check{H}^1(K) \neq \mathbb{Z}$

Theorem

Let K be a non-trivial toroidal set with $\check{H}^1(K) \neq \mathbb{Z}$ then either K is unknotted or $g(K) = \infty$.

Proof.

Suppose that $g(K)$ is finite and consider a nested basis of solid tori $\{T_n\}_{n\geq 0}$. Then the sequence of genera $\{g_n\}_{n\geq 0}$ is constant for n sufficiently large. As a consequence, for every i large we may assume that $w_i \neq 0, 1$ and if we fix $j > i$ we get

$$g_i = g_{j+1} \geq w_j \cdot w_{j-1} \cdot \ldots \cdot w_{i+1} \cdot w_i \cdot g_i.$$

Therefore, either $w_j \cdot w_{j-1} \cdot \ldots \cdot w_{i+1} \cdot w_i \leq 1$ or $g_i = 0$. Since the first possibilty is excluded we get $g(K) = 0$. □

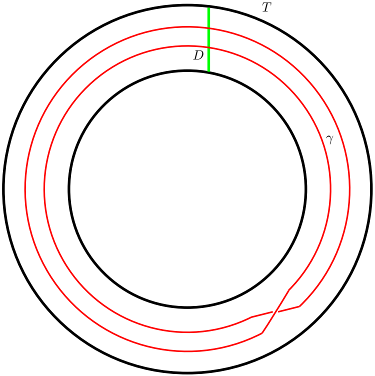
Solenoids that cannot be attractors

Corollary

Any solenoid can be embedded in $\mathbb{R}^3$ in uncountably many inequivalent ways such that none of them can be realized as an attractor for a homeomorphism of $\mathbb{R}^3$.

What happens in the case of finite genus?

In the case of finite genus we shall use the *geometric index* of a loop inside a torus defined by Schubert (1953). Let $T \subset \mathbb{R}^3$ be a solid torus and $\gamma \subset \mathrm{int}\ T$ a closed curve (both polyhedral). Consider a meridional disk D of T that intersects γ transversally.



Geometric index

Definition

The *geometric index* of γ in T is defined as the minimum number of intersections of γ with all the meridional disks of T that intersect γ transversally.

Geometric index of solid tori

The notion of geometric index just defined is easily extended to the case of two (polyhedral) solid tori $T' \subset \mathrm{int}\ T$ in the obvious way, that is:

Definition

The geometric index of T' in T is defined as the geometric index of a core of T' in T and is denoted by $N(T', T)$.

Multiplicativity of the geometric index

An important property of the geometric index is the following:

Property

If $T'' \subset T' \subset T$ are nested (polyhedral) solid tori. Then

$$N(T'', T) = N(T'', T') \cdot N(T', T).$$

Geometric index and nested bases of neighborhoods

Let K be a toroidal set and $\{T_n\}_{n\geq 0}$ be a nested basis of neighborhoods comprised of solid tori. If we denote by $N_j = N(T_{j+1}, T_j)$ the multiplicativity property just mentioned allows us to consider following direct sequence:

$$\mathbb{Z} \xrightarrow{\cdot N_1} \mathbb{Z} \longrightarrow \cdots \xrightarrow{\cdot N_j} \mathbb{Z} \longrightarrow \cdots \qquad (1)$$

Geometric index and nested bases of neighborhoods

Proposition

The direct limit of (1) *is independent of the chosen basis.*

Self-geometric index of a toroidal set

Definition

Let K be a toroidal set. We define the *self-geometric index* of K denoted by $\mathcal{N}(K)$ as the direct limit of (1) for any basis $\{T_j\}$ of K comprised of nested (polyhedral) solid tori.

Notice that every toroidal set has non-trivial self-geometric index.

Feasible groups

We call *feasible* to any group G obtained as a direct limit of a direct sequence of the form

$$\mathbb{Z} \xrightarrow{\cdot m_1} \mathbb{Z} \longrightarrow \cdots \xrightarrow{\cdot m_j} \mathbb{Z} \longrightarrow \cdots$$

Types of feasible groups

1. $G = 0$ if and only if $m_j = 0$ for infinitely many j.
2. $G = \mathbb{Z}$ if and only if $m_j = 1$ for j sufficiently large.
3. Otherwise G is not finitely generated.

Self-geometric index and flows

The self-geometric index together with the genus is sufficient to solve the realizability in the case of flows:

Theorem

Let $K \subset \mathbb{R}^3$ be a toroidal set. Then K can be realized as an attractor for a flow if and only if

1. *K has finite genus and*
2. *$\mathcal{N}(K) = \mathbb{Z}$.*

In the case of non-trivial toroidal sets we have the following somewhat surprising property:

Proposition

Let K be a non-trivial toroidal set ($\check{H}^1(K) \neq 0$) such that $0 < g(K) < \infty$. Then $\mathcal{N}(K) = \mathbb{Z}$.

Realization problem for non-trivial toroidal sets of positive finite genus

Corollary

Let K be a knotted non-trivial toroidal set. Then K can be realized as an attractor of a flow if and only if its genus is finite.

Number-like feasible groups

In the particular case whenever $m_j = m \geq 1$ for j sufficiently large

$$G \cong \begin{cases} \mathbb{Z} & \text{if } m = 1 \\ \mathbb{Z}\left[\frac{1}{m}\right] & \text{otherwise} \end{cases}$$

and G is said to be *number-like*.

Self-geometric index of a toroidal attractor

Theorem

Let K be a toroidal set that is an attractor for a homeomorphism f of $\mathbb{R}^3$. Then $\mathcal{N}(K)$ is number-like. Moreover, $\mathcal{N}(K) = \mathbb{Z}$ if and only if K can be realized as an attractor for a flow.

Idea of the proof

The first part of the proof uses the fact that the geometric index is an invariant the pair (T, T'). The second part uses that since K is an attractor of a homeomorphism it has finite genus.

Realization problem for solenoids

Corollary

A solenoid can be realized as an attractor for a homeomorphism if and only if it is an unknotted n-adic solenoid.

Winding number and geometric index

It is not difficult to see that if $T' \subset T$ is a pair of nested (polyhedral) solid tori w denotes the winding number and N the geometric index of T' in T respectively we have the following relations

1. $0 \leq w \leq N$.
2. $N \equiv w \mod 2$.

Self-geometric index and Čech cohomology

A consequence of the previous relations is that if H and N are feasible groups and K is a toroidal set such that $\check{H}^1(K) = H$ and $\mathcal{N}(K) = N$ then they satisfy the two following compatibility conditions

(C1) $2 \mid H$ if and only if $2 \mid N$.

(C2) If $N = \mathbb{Z}$ then $H = \mathbb{Z}$.

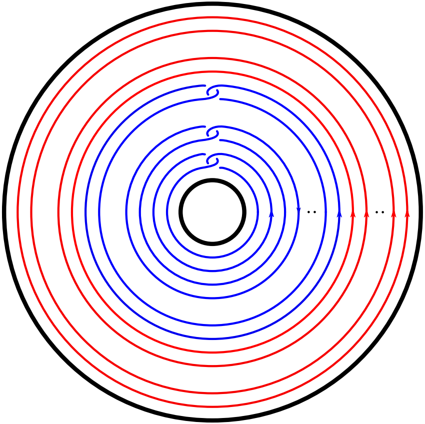
The notation $2 \mid G$ means that for every element $z \in G$ there exists some $z' \in G$ such that $z = 2 \cdot z'$.

A useful lemma

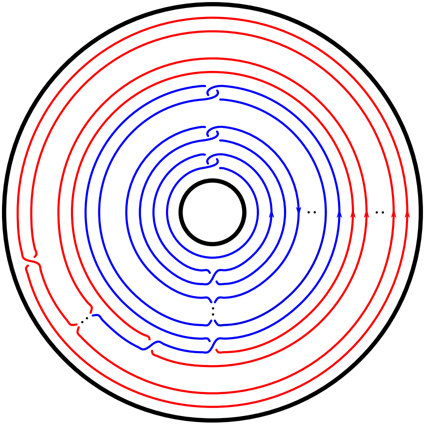
Lemma

Let w and k be a pair of non-negative integers. Denote by $V_0 \subset \mathbb{R}^3$ the standard unknotted solid torus in $\mathbb{R}^3$. Then there exists and unknotted solid torus $V_1 \subset \operatorname{int} V_0$ such that the winding number of V_1 in V_0 is w and the geometric index $N(V_1, V_0) = w + 2k$.

Idea of the construction



Idea of the construction



Realization of feasible groups as self-indices and Čech cohomology groups of unknotted toroidal sets

Theorem

Let H and N be feasible groups. Suppose that $N \neq 0$, and H and N satisfy (C1) and (C2). Then there exists an unknotted toroidal set K with $\check{H}^1(K) = H$ and $\mathcal{N}(K) = N$.

Unknotted toroidal sets that cannot be attractors

Theorem

Let H be a feasible group. Then there exists an uncountable family $\{K_\alpha\}$ of toroidal sets such that:

1. *None of the K_α can be realized as an attractor for a homeomorphism of $\mathbb{R}^3$.*
2. *The K_α are pairwise different (not ambient homeomorphic).*
3. *Each K_α is unknotted.*
4. *Each K_α has H as its first Čech cohomology group.*

In summary

So far we have seen that there are the following possibilities for toroidal sets in terms of the genus g:

1. Infinite genus toroidal sets cannot be attractors.
2. Finite positive genus non-trivial toroidal sets are attractors of flows.
3. There are plenty of examples of unknotted toroidal sets that cannot be attractors.

What happens with finite positive genus trivial toroidal sets?

Theorem

Let $N \neq 0$ be a feasible group such that $2 \mid N$. Then there exists a toroidal set K such that $\mathcal{N}(K) = N$ and $g(K) = 1$.

Of course if we choose N in such a way that it is not number-like, then K cannot be realized as an attractor. It is clear that K must be trivial. The construction makes use of the Whitehead double construction.

References

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Thank you very much for your attention!