Connections between van der Corput's Difference Theorem and the Ergodic Hierarchy of Mixing Properties

Dynamical Systems Seminar
at Jagiellonian University in Kraków

Based on https://arxiv.org/abs/2106.01123

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March 18, 2022

The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is uniformly distributed if for any open interval $(a, b) \subseteq [0, 1]$ we have

$$\lim_{N\to\infty} \frac{1}{N}\Big|\{1 \le n \le N \mid x_n \in (a, b)\}\Big| = b - a. \tag{1}$$

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Theorem (van der Corput)

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n \pmod 1)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

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Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed.

Theorem (HvdCDT1)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} ||\frac{1}{N} \sum_{n=1}^{N} x_n|| = 0. \tag{3}$$

Theorem (HvdCDT2)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty}\left|\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0,, \tag{4}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0. \tag{5}$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ *is a bounded sequence satisfying*

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \left| \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{6}$$

then

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Theorem (HvdCDT3)

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Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system $(X, \mathscr{B}, \mu, T)$, *and any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu(A \cap T^{-n}A) > 0. \tag{8}$$

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Furstenbergs proof uses HvdCDT1.

Applications of HvdCDTs 2/2

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Furstenberg's proof uses an equivalent form of HvdCT3.

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Furstenberg's proof uses an equivalent form of HvdCT3. Other proofs directly use HvdCT3.

The Ergodic Hierarchy of Mixing 1/2

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Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a measure preserving system. If for every $A, B \in \mathscr{B}$ we have

1. $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$, then $\mathcal{X}$ is ergodic.

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The Ergodic Hierarchy of Mixing 2/2

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Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a measure preserving system. If there exists a σ-algebra $\mathscr{A}$ such that $\{T^{-n}A \mid A \in \mathscr{A}, n \geq 0\}$ generates $\mathscr{B}$, and for every $A, B \in \mathscr{A}$ and $n \geq 1$ we have $\mu(A \cap T^{-n}B) = \mu(A)\mu(B)$, then $\mathcal{X}$ is Bernoulli.

Symmetry and Mixing 1/2

Theorem

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a measure preserving system. If for every $A \in \mathscr{B}$ we have

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Hilbertian (Cesàro) vdCDTs Revisited

Theorem

Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequences. If

$$(i) \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0 \text{ for every } h \in \mathbb{N}, \text{ or}$$

$$(ii) \lim_{h\to\infty} \overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \text{ or}$$

$$(iii) \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \text{ then } \lim_{N\to\infty} \left|\left|\frac{1}{N}\sum_{n=1}^{N} x_n\right|\right| = 0.$$

Theorem (MvdCDT1)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ *is a bounded sequence satisfying*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} \langle x_{n+h}, x_n \rangle = 0, \tag{11}$$

for every $h \in \mathbb{N}$, *then* $(x_n)_{n=1}^{\infty}$ *is a* **nearly orthogonal sequence**.

Context of Nearly Mixing Sequences

One way to understand this result is to consider a new Hilbert space $\mathcal{H}'$, whose elements are sequences $(x_n)_{n=1}^{\infty}$ of vectors coming from $\mathcal{H}$.

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$$\langle (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \rangle_{\mathcal{H}'} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, y_n \rangle \tag{12}$$

be the inner product on $\mathcal{H}'$.

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be the inner product on $\mathcal{H}'$. The hypothesis that

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} \langle x_{n+h}, x_n \rangle = \langle U^h (x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty} \rangle_{\mathcal{H}'}, \tag{13}$$

$$(cf. \mu(A \cap T^{-n} A) = \mu(A)^2 \ \forall \ A \in \mathscr{A}, n \geq 1)$$

for every $h \in \mathbb{N}$ verifies that $\{U^h (x_n)_{n=1}^{\infty}\}_{h=0}^{\infty}$ is an orthonormal set in $\mathcal{H}'$, where U denotes the left shift operator.

Strong Mixing vdCDT

Theorem (MvdCDT2)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle\right| = 0, \tag{14}$$

then $(x_n)_{n=1}^{\infty}$ is a **nearly strongly mixing sequence**.

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Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before.

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then $(x_n)_{n=1}^{\infty}$ *is a* **nearly strongly mixing sequence**.

Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before. The given hypothesis implies

$$0 = \lim_{h\to\infty}\langle U^h(x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}\rangle_{\mathcal{H}'}, \tag{15}$$

$$\text{(cf. } \lim_{h\to\infty}\mu(A\cap T^{-n}A) = \mu(A)^2\ \forall\ A\in\mathscr{B})$$

verifies that $\{U^h(x_n)_{n=1}^{\infty}\}_{h=0}^{\infty}$ is a strongly mixing sequence in $\mathcal{H}'$.

Weak Mixing vdCDT

Theorem

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ *is a bounded sequence satisfying*

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \left| \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{\infty} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{16}$$

then $(x_n)_{n=1}^{\infty}$ *is a* **nearly weakly mixing sequence**.

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Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before.The given hypothesis implies

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$$\left(\text{cf. } \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\mu(A\cap T^{-n}A) - \mu(A)^2| = 0\ \forall\ A\in\mathscr{B}\right)$$

verifies that $\{U^h(x_n)_{n=1}^\infty\}_{h=0}^\infty$ is a weakly mixing sequence in $\mathcal{H}'$.

Properties of Nearly Weakly Mixing Sequences

Theorem

Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a nearly weakly mixing sequence, $(r_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a compact sequence, and $(c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ a compact sequence.

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, r_n \rangle = 0 \tag{18}$$

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Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a nearly strongly mixing sequence, $(r_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a rigid sequence, and $(c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ a rigid sequence.

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A Question of Frantzikinakis

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathscr{B}, \mu, T)$ has zero entropy and $f, g \in L^\infty(X, \mu)$.

(i) Is it true that the averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{p(n)} g \tag{22}$$

converge in $L^2(X, \mu)$ when $p(n) = n$ or $p(n) = n^2$?

(ii) Is it true that for every $A \in \mathscr{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n} A \cap S^{-p(n)} A) > 0 \tag{23}$$

when $p(n) = n$ or $p(n) = n^2$?

Applying MvdCDTs 1/3

Theorem

Let $(X, \mathscr{B}, \mu)$ *be a probability space and let* $T, S : X \to X$ *be measure preserving transformations. Suppose that the m.p.s.* $(X, \mathscr{B}, \mu, T)$ *is rigid, and that the m.p.s.* $(X, \mathscr{B}, \mu, S)$ *is totally ergodic. Let* $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ *be a sequence for which* $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ *is uniformly distributed for every* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Applying MvdCDTs 1/3

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Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathscr{B}, \mu, T)$ is rigid, and that the m.p.s. $(X, \mathscr{B}, \mu, S)$ is totally ergodic. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

1. *If $f, g \in L^{\infty}(X, \mu)$ are such that $\int_X g d\mu = 0$, then $(T^n f \cdot S^{k_n} g)_{n=1}^{\infty}$ is a nearly weakly mixing sequence in $L^2(X, \mu)$.*

Applying MvdCDTs 2/3

Theorem

(ii) *For any $f, g \in L^\infty(X,\mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T] \int_X g d\mu, \tag{24}$$

where $\mathcal{I}_T = \{A \in \mathscr{B} \mid T^{-1}A = A\}$ is the σ-algebra of T-invariant sets and with norm-convergence.

Applying MvdCDTs 3/3

Theorem

(iii) *If* $A_1, A_2, A_3 \in \mathscr{B}$ *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_1 \cap T^{-n}A_2 \cap S^{-k_n}A_3)$$
$$= \mu(A_3) \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_1 \cap T^{-n}A_2).$$

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(iii) *If* $A_1, A_2, A_3 \in \mathscr{B}$ *then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A_1\cap T^{-n}A_2\cap S^{-k_n}A_3)$$
$$=\mu(A_3)\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A_1\cap T^{-n}A_2).$$

(iv) *If* $((k_{n+h}-k_n)\alpha)_{n=1}^{\infty}$ *is uniformly distributed in its orbit closure for all* $\alpha\in\mathbb{R}$ *then (i)-(iii) hold when* $(X,\mathscr{B},\mu,S)$ *is ergodic.*

Applications to Uniform Distribution

See Section 4 in https://arxiv.org/abs/2106.01123 for applications to uniform distribution.