Chaotic Dynamics for Maps in One and Two Dimentions

The method of Stretching Along the Paths (SAP)

The SAP method consists in proving the presence of chaotic dynamics in the sense of Theorem 2 by verifying the SAP property (according to Definition of SAP Property) with respect to some disjoint compact sets K_0 and K_1, which in turn establishes the existence of the conditions of Theorem 3.

A path in a metric space X is a continuous map $\gamma : [t_0; t_1] \longrightarrow X$. We also set. Without loss of generality, we usually take the unit interval $[0, 1]$ as the domain of γ. A sub-path ω of γ is the restriction of γ to a compact sub-interval of its domain. An arc is the homeomorphic image of the compact interval $[0, 1]$. By a generalized rectangle we mean a set $\mathcal{R} \subset X$ which is homeomorphic to the unit square $[0, 1]^2 \subset \mathbb{R}^2$. Given a generalized rectangle $\mathcal{R}$ and the associated homeomorphism $h : [0, 1]^2 \longrightarrow h([0, 1]^2) = \mathcal{R}$, the set $con\mathcal{R} := h(\partial([0, 1]^2))$, is named the contour of $\mathcal{R}$. We also call oriented rectangle the pair

$$\tilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-),$$

where $\mathcal{R} \subset X$ is a generalized rectangle and

$$\mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^-$$

is the union of two disjoint compact arcs $\mathcal{R}_l^-, \mathcal{R}_r^- \subset con\mathcal{R}$ that we call the left and the right sides of $\mathcal{R}^-$.

Definition (SAP).
Suppose that $\psi : X \supset D_\psi \longrightarrow X$ is a map defined on a set D_ψ and let $\tilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\tilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles of a metric space X. Let $K \subset \mathcal{A} \cap D_\psi$ be a compact set. We say that (K, ψ) strechtes $\tilde{\mathcal{A}}$ to $\tilde{\mathcal{B}}$ along the paths and write:

$$(K, \psi) : \tilde{\mathcal{A}} \rightsquigarrow \tilde{\mathcal{B}}$$

if the following conditions hold:
⋄ ψ is continous on K
⋄ for every path $\gamma : [0, 1] \longrightarrow \mathcal{A}$ such that $\gamma(0) \in \mathcal{A}_l^-$ and $\gamma(1) \in \mathcal{A}_r^-$ (or $\gamma(1) \in \mathcal{A}_l^-$ and $\gamma(0) \in \mathcal{A}_r^-$), there exists a sub-interval $[t', t''] \subset [0, 1]$ such that

$$\gamma(t) \in K,\ \psi(\gamma(t)) \in \mathcal{B},\ \forall t \in [t', t'']$$

and, moreover, $\psi(\gamma(t'))$ and $\psi(\gamma(t''))$ belong to different components of $\mathcal{B}^-$.

Theorem 1
Let $\psi : X \supset D_\psi \longrightarrow X$ be a map defined on a set D_ψ and let $\tilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle of a metric space X. If $K \subset \mathcal{R} \cap D_\psi$ is a compact set for which

$$(K, \psi) : \tilde{\mathcal{R}} \rightsquigarrow \tilde{\mathcal{R}}$$

then there exists at least a point $z \in K$ with $\psi(z) = z$.

Definition.
Let X be a metric space, $\psi : X \supset D_\psi \longrightarrow X$ be a map and let $D \subset D_\psi$. We say that ψ *induces chaotic dynamics on the set* D if there exist two nonempty disjoint compact sets

$$K_0,\ K_1 \subset D$$

such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in D^{\mathbb{Z}}$ such that

$$w_i \in K_{s_i} \textit{ and } w_{i+1} = \psi(w_i),\ \forall i \in \mathbb{Z}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i, \ \forall i \in \mathbb{Z}$ for some $k \geqslant 1$, there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in D^{\mathbb{Z}}$ satisfying: $w_i \in K_{s_i} \ and \ w_{i+1} = \psi(w_i), \ \forall i \in \mathbb{Z}$.

Theorem 2
Let ψ be a map inducing chaotic dynamics on a set D and which is continuous on

$$K := K_0 \cup K_1 \subset D$$

where $K_0, \ K_1, \ D$ are as in Definition above. Defining the nonempty compact set

$$\mathcal{I}_\infty := \bigcap_{n=0}^{\infty} \psi^{-n}(K),$$

then there exists a nonempty compact set

$$\mathcal{I} \subset \mathcal{I}_\infty \subset K,$$

on which the following are fulfilled:
(i) $\mathcal{I}$ is invariant for ψ
(ii) $\psi|_{\mathcal{I}}$ is semi-conjugate to the Bernoulli shift on two symbols, that is, there exists a continuous map π of $\mathcal{I}$ onto $\sum_2^+ := \{0,1\}^{\mathbb{N}}$, endowed with the distance

$$\hat{d}(s', s'') := \Sigma_{i\in\mathbb{N}} \frac{d(s'_i, s''_i)}{2^{i+1}}, \ for \ s' = (s'_i)_{i\in\mathbb{N}}, \ s'' = (s''_i)_{i\in\mathbb{N}} \in \Sigma_2^+$$

(where $d(\ , \ )$ is the discrete distance on $\{0,1\}$ such that $d(s'_i, s''_i) = 0$ for $s'_i = s''_i$ and $d(s'_i, s''_i) = 1$ for $s'_i \neq s''_i$), such that the diagram commutes, where $\sigma : \Sigma_2^+ \longrightarrow \Sigma_2^+$ is the Bernoulli shift defined

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathcal{I} \arrow[d, "\pi"] \\ \Sigma_2^+ \arrow[r, "\sigma"'] & \Sigma_2^+ \end{tikzcd}$$

by $\sigma((s_i)_i) = (s_{i+1})_i$
(iii) The set $\mathcal{P}$ of the periodic points of $\psi|_{\mathcal{I}_\infty}$ is dense in $\mathcal{I}$ and the pre-image $\pi^{-1}(s) \subset \mathcal{I}$ of every k-periodic sequence $s = (s_i)_{i\in\mathbb{N}} \in \Sigma_2^+$ contains at least one k-periodic point.
Furthermore, from property (ii) it follows that:
(iv)

$$h_{top}(\psi) \geqslant h_{top}(\psi|_{\mathcal{I}}) \geqslant h_{top}(\sigma) = log(2)$$

where h_{top} is the topological entropy.
(v) There exists a compact invariant set $\Gamma \subset \mathcal{I}$ such that $\psi|_\Gamma$ is semi-conjugate to the Bernoulli shift on two symbols, topologically transitive and has sensitive dependence on initial conditions.

Theorem 3
Let $\tilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle of a metric space X and let $D \subset \mathcal{R} \cap D_\psi$, with D_ψ the domain of a map $\psi : X \supset D_\psi \longrightarrow X$. If K_0 and K_1 are two disjoint compact sets with $K_0 \cup K_1 \subset D$ and

$$(K_i, \psi) : \tilde{R} \rightsquigarrow \tilde{R}, \ for \ i = 0, 1$$

then ψ induces chaotic dynamics on two symbols on D.

It follows that the map ψ possesses the properties $(i) - (v)$ of Theorem 2.