A symmetry of maps implies its chaos, i.e. gives ∞ many periodic points

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Notation:

$P^n(f) := \mathrm{Fix}\,(f^n)$ **the set of points of period n**,

$$P_n(f) := P^n(f) \setminus \bigcup_{k|n<n} P^k(f)$$

the set of points for which n is the **minimal period**, called shortly n-**periodic** points.

$$P(f) := \bigcup_{1}^{\infty} P^n(f) = \bigcup_{1}^{\infty} P_n(f)$$

the set of **all periodic points**.

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Theorem (Shub & Sullivan 1974)

$$f : X \to X \quad \textit{such that} \quad \{L(f^n)\}_1^\infty \quad \textit{is unbounded}$$

$$\textit{If} \quad f \in C^1 \quad \textit{then} \quad \#P(f) = \infty$$

Lefschetz-Hopf formula

$$L(f^n) = \mathrm{Ind}(f^n, X) = \sum_{x \in \mathrm{Fix}\,(f^n)} \mathrm{Ind}(f^n, x)$$

Main step: $f \in C^1, f(0) = 0 \Longrightarrow \{\mathrm{Ind}(f^n, 0)\}$ is bounded provided it is defined.
Chow & Mallet-Paret & Yorke (81): the sequence is periodic of a period k defined by $\sigma(Df(0))$.

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Conjecture (Shub 1974)

$$\limsup \sqrt[n]{\#P^n(f)} \geq \limsup \sqrt[n]{|L(f^n)|} = \rho_{es} > 1 \ \ \textit{if}\{L(f^n)\} \textit{ is unb.}$$

The rate of growth is at least exponential.

Theorem (Babenko & Bogatyi 1991)

$f : X \to X$, $d = \dim X$, *and* $f \in C^1$. *Assume:* $\{L(f^n)\}_1^\infty$ *is unb.* *Then* $\exists\, n_0 = n_0(f)$ *such that* $\forall\, n \geq n_0$

$$\#\mathrm{Or}(f, n) \geq \frac{n - n_0}{D\, 2^{[d+1/2]}},$$

where $D := \dim H^*(X; \mathbb{R})$

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Theorem ([JJWM1])

Let $g : S^d \to S^d$, $d \geq 1$, be a free homeomorphism of finite order $m > 1$, and $f : S^d \to S^d$ be a map that commutes with g. Suppose that $\deg(f) \notin \{-1, 0, 1\}$. Then for $\forall\ k \in \mathbb{N}$ we have

$$\#\mathrm{Fix}\,(f^{km}) \;\geq\; m^2\, k'$$

k' is as in Definition [19]. In particular, for $k = m^s$ we have

$$\#\mathrm{Fix}\,(f^{m^{s+1}}) \;\geq m^{s+2}\,.$$

For S^d, $d \geq 1$, $\{L(f^k)\}_1^\infty$ is unbounded iff $\deg(f) \neq 0, \pm 1$.

Corollary ([JJWM1])

Let $f : S^d \to S^d$ be a continuous map with $\{L(f^n)\}_1^\infty$ unbounded. If f commutes with a free homeomorphism $g : S^d \to S^d$ of order $m > 1$, then the set $\mathrm{Per}(f)$ of minimal periods of f and consequently the set $P(f)$ of periodic points are infinite.

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Eventually generalize for G-equivariant maps? for G not cyclic, or infinite compact

1. Few class of groups: There exists a classification of finite groups which can act free (act smoothly) on the spheres!!, e.g. they do not contain $\mathbb{Z}_p \oplus \mathbb{Z}_p$.
2. The only infinite groups acting free on S^d : $G = S^1$, $N(S^1) \subset S^3$, S^3 .
3. If $f : M \to M$ is G-equivariant, M any compact manifold, G infinite compact, then $L(f) = 0$, consequently $L(f^k) = 0$ for $\forall \ k$.
4. For the problem of existence of infinitely many periodic points, one can always restrict the action to a cyclic subgroup of G.

Remark

Note that in the the simplest case $G = \mathbb{Z}_2$, or $G = \mathbb{Z}_m$ the orbit space S^d/G is the real projective space, or a lens space.

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The idea of proof

- The general idea of the proofs of the stated Theorems is to study a map $\bar{f} : M \to M$ of the quotient space $M := S^d/\mathbb{Z}_m$ induced by the $\mathbb{Z}_m$-equivariant map $f : S^d \to S^d$ in the problem.
- Next we estimate the number of periodic points of $\bar{f}$, and we "lift" them to periodic points of f.
- To study periodic points of the induced map $\bar{f}$ we use the Nielsen theory adapted to this situation.
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Below we include some facts about equivariant maps

Proposition (Borsuk-Ulam)

Suppose that $\mathbb{Z}_m$ acts freely on S^d, $d \geq 1$. If $f : S^d \to S^d$ is an equivariant map, then $\deg(f) \equiv 1 \mod m$. □

For $m = 2$, this is the classical Borsuk-Ulam theorem which states that **an odd map has odd degree** .
$f, h : S^d \to S^d$ are homotopic $\iff \deg f = \deg h$.

Theorem (C, Bowszyc, R. Rubinsztein, [Rub].)

Suppose that a finite group G acts freely on S^d, $d > 1$.
Then the natural map $[S^d, S^d]_G \to [S^d, S^d]$ is an injection, i.e. if two equivariant mappings have the same degree, then they are **equivariantly homotopic**.
Moreover the image of $[S^d, S^d]_G$ in $[S^d, S^d] = \mathbb{Z}$ is equal to $\{m\mathbb{Z} + 1\}$.

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Comments

Note that taking the suspension of the map $f : S^1 \to S^1$, $f(z) = z^r$, $|r| \geq 2$ (with ∞ many periodic points) we get a map Σf of S^2 with the same dynamics as z^r.

On the other hand, the Shub example gives a map of S^2 which is a small perturbation of Σf but has only two non-wandering points.

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In this section we denote by $p : \tilde{X} \to X$ a universal covering of X.

For a G-space X with a free action of a finite group G and a map $X \to \bar{X} = X/G$ onto the orbit space we will denote the covering $p : X \to \bar{X}$, opposite to the notation used here..

Let $p : \tilde{X} \to X$ be a universal covering of a polyhedron.

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Let $f : X \to X$ be a map and let $\mathrm{lift}(f) = \{\tilde{f} : \tilde{X} \to \tilde{X} : \ p\tilde{f} = fp\}$ denote *the set of all lifts* of f.

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Theorem

For any self-map $f : X \to X$ of a finite polyhedron and a fixed natural number $k \in \mathbb{N}$

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Corollary

Let $\bar{f} : M \to M$ be the map induced by an equivariant map $f : S^d \to S^d$ of degree $\neq 0, \pm 1$. Then all the Reidemeister classes of $\bar{f}$ and of all its iterations are essential.

IT is A KEY POINT which uses the fact that $M = S^d/G$. In general we need an information that

$$L(f) \neq 0 \Longrightarrow \forall g \in G,\, g \neq e, \text{ we have } L(gf) \neq 0$$

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The Reidemeister relation of the map $\bar{f} : \bar{X} \to \bar{X}$ induced by an equivariant map $f : X \to X$ is trivial. Thus $\mathcal{R}(\bar{f}) = \mathcal{O}_{\bar{X}} = \mathbb{Z}_m$.

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Thus it remains to find the number of irreducible classes in $\mathcal{R}(\bar{f}^r)$ for such r (or for every r). Let us recall that the class $A \in \mathcal{R}(\bar{f}^k)$ is reducible iff it belongs to the image of the map $i_{kl} : \mathcal{R}(\bar{f}^l) \to \mathcal{R}(\bar{f}^k)$ for an $l \mid k$, $l < k$.

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Equivariant Nielsen theory for a free action on arbitrary M

Theorem ([JJWM2])

Let M be a finite polyhedron with a free action of a finite group G and $f : M \xrightarrow{G} M$ an equivariant map. Then $\exists$ an invariant
$$NF_n^G(f) \in \{0\} \cup \mathbb{N} \text{ such that}$$

1. $NF_n^G(f)$ *is a G-homotopy invariant.*
2. $\#\mathrm{Fix}(g^k) \geq NF_n^G(f)$ *for each* $g \sim_G f$.

Let $G = \mathbb{Z}_{p^a}$ where p is a prime and let $n = p^\alpha$.

Theorem ([JJWM2])

If for every $k \in \mathbb{N}$ all the Reidemeister classes of $[\bar{f}^k]$, where $\bar{f} = f/G$, are essential then

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Dependence of $k \mapsto N_k^G(f)$ on m

Definition 19

We say that a natural number r *eventually divides* m if r divides a power m^s. In other words r eventually divides m if and only if for a prime number p

$$p|r \Rightarrow p|m$$

We define k' as the greatest divisor of k dividing eventually m.

Theorem

Let G be a finite abelian group, $\#G = m$. Then

$$NF_k(f) = NF_{k'}(f)$$

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Assumption

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Let $G = \pi_1 X = \mathbb{Z}_{p_1^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{a_r}}$, where $p_1, \ldots, p_r$ denote different primes, be a cyclic group of order $m = p_1^{a_1} \cdots p_r^{a_r}$. Then for k eventually dividing m

$$NF_k^G(f) = \begin{cases} km & \text{if } m|k \\ \gcd(m,k) \cdot m & \text{otherwise} \end{cases}$$

If k does not event. divides m then $NF_k^G(f) = 0$ $\square$

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We recall that $\mathcal{R}(\bar{f}^k) = \mathbb{Z}_p$ and $i_{p^\alpha,p^\beta}[x] = [p^{\alpha-\beta} \cdot x]$. This implies that:

1. all classes in $\mathcal{OR}(\bar{f}^1) = \mathcal{R}(\bar{f}^1) = \mathbb{Z}_p$ are irreducible while for $\alpha \geq 1$
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Under the above assumptions (and $\alpha \geq 1$)

$$NF_{p^\alpha}^{\mathbb{Z}_p}(f) = p + \sum_\beta (p^{\beta+2} - p^{\beta+1})$$

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$$L(f^{p^a}) \not\equiv 0 \mod (p^{a+1})$$

then f has infinitely many periodic points and

$$\limsup \frac{\#\mathrm{Fix}(f^n)}{n} \geq p\,.$$

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