Estimates of the topological entropy from below for continuous self-maps on some compact manifolds

Wacław Marzantowicz & Feliks Przytycki

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Let M be a compact manifold
ρ be a metric on M consistent with the topology.
$f : M \to M$ a continuous self-map of M.

Definition

The topological entropy, denoted by $\mathbf{h}(f)$, is defined as

$$\lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} 1/n \log \sup \# Q,$$

the supremum over all Q being (ϵ, n)-separated.
Q is called (ϵ, n)-separated, if for every two distinct points $x, y \in Q$,

$$\max_{j=0,\ldots,n} \rho(f^j(x), f^j(y)) \geq \epsilon .$$

In fact $\mathbf{h}(f)$ does not depend on the metric (cf. [Szl]).
Warning!: $\mathbf{h}(f)$ is not continuous in C^0-topology,
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Entropy Conjecture

Entropy Conjecture, denoted shortly as **EC**, says that

$$\log \mathrm{sp}(f) := \log \mathrm{sp}\left(H_*(f)\right) \leq \mathbf{h}(f)\,.$$

It was posed by M. Shub in seventies who asked what suppositions on f or M imply EC.

Meantime a few results have been shown:

f is C^1: $\log|\deg(f)| \leq \mathbf{h}(f)$ (Misiurewicz-Przytycki),

f is C^0: $\log \mathrm{sp}(H_1(f)) \leq \mathbf{h}(f)$ (Manning),

f is C^∞: **EC** holds (Yomdin),

$M = S^d$ then **EC** is not true for $f \in C^0$,

f is C^0: $M = \mathbb{T}^d$ the torus **EC** holds (Misiurewicz-Przytycki)

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Remark (Anatoly Katok conjecture - 1978)

A conjecture [Ka] saying that
EC holds for every continuous map if the universal cover *of M is* homeomorphic *to an Euclidean space* $\mathbb{R}^d$.

Let π be a (discrete) group, $F : \pi \to \pi$ a homomorphism. M be a manifold, dim $M = d$, which is $K(\pi, 1)$ and a continuous map

$$f : M \to M \quad \text{such that the induced map} \quad f_{\#} = F\,.$$

Does there exist a numeric invariant $\mathbf{inv}(F) \in \mathbb{R}$ which

1. estimates from below the topological entropy of f,
   i.e. $\mathbf{inv}(f_{\#}) \leq \mathbf{h}(f)$.
2. it is the best estimate in a homotopy class,
   i.e. $\exists$ a map $f' \sim f$ such that $\mathbf{inv}(f_{\#}) = \mathbf{h}(f')$.

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Logarithmic growth of a homomorphism

Definition

Let $\gamma \in \pi$-finitely generated group, with the set of generators $\mathcal{S}$.
$\gamma = \gamma_1^{a_1} \cdots \gamma_s^{a_s} \gamma_1^{a_{s+1}} \cdots \gamma_s^{a_{2s}} \cdots \gamma_s^{a_{k_s}}$, $a_j \in \mathbb{Z}$.

$L(\gamma, \mathcal{S}) := \min\limits_{\text{present.}} \sum_{j=1}^{k_s} |a_j|$. For a homomorphism $F : \pi \to \pi$

$L(F, \mathcal{S}) := \max\limits_{1 \le i \le s} L(F(\gamma_i), \mathcal{S}), \quad \mathbf{h}_{\mathcal{AL}}^{\mathcal{S}}(F) := \lim_{n\to\infty} \frac{1}{n} \log L(F^n, \mathcal{S})$.

Theorem (Manning)

$\mathbf{h}_{\mathcal{AL}}^{\mathcal{S}}(F)$ does not depend on $\mathcal{S}$, $\mathbf{h}_{\mathcal{AL}}(f_\#) = \mathbf{h}_{\mathcal{AL}}^{\mathcal{S}}(f_\#)$ and $\le \mathbf{h}(f)$ for $f : M \to M$, $f_\# : \pi_1(M) \to \pi_1(M)$, M any manifold.

Remark

If $F : \mathbb{Z}^d \to \mathbb{Z}^d$ is given by a matrix A then $\mathbf{h}_{\mathcal{AL}}(F) = \log \operatorname{sp}(A)$.

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π virtually nilpotent if it contains a finite index nilp. subgr. $\Gamma \subset \pi$

We can assume that a nilp. finite ind. subgr. $\Gamma \lhd \pi$ i.e. is normal.

Remark (1)

$\Gamma \subset G$ a lattice, i.e. a **discrete co-compact** *subgroup, of a* **connected, simply con. nilpotent** *Lie group G.*

π is fininitely gen. torsion free as a $\pi_1(K(\pi, 1))$.

Remark (2)

π is virtually nilpotent $\Longleftrightarrow$ π has polynomial growth, *[Gr1].*

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If $\mathbf{C} \subset \mathrm{Aut}(G)$ - the maximal compact subgroup, then $\pi \subset G \ltimes \mathbf{C} \subset G \ltimes \mathrm{Aut}(G)$; this embedding of π is called an almost Bieberbach group.

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By definition π acts on G properly discontinuously.

Definition

The quotient $\mathbf{IN} = G/\pi$ is called an *infra-nilmanifold*.

$\mathbf{IN}$ is regularly finitely covered by $\mathbf{N} = G/\Gamma$,
with the deck transformation group equal to $H = \pi/\Gamma$.

Thus $\mathbf{IN}$ is a $K(\pi,1)$ manifold.

The **image** of π into $G \ltimes \mathbf{C} \subset \mathrm{Aut}(G)$ we denote by $\pi_{\mathbf{IN}}$.

Remark (4)

*Similarly to $\mathbf{N} \to \mathbf{IN} = G/\pi_{\mathbf{IN}}$, we have $\tilde{M}/\Gamma \to M = \tilde{M}/\pi$, with the **universal cover** $\tilde{M}$ and the **deck transformation group** H.*

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Theorem (EC for virtually nilpotent $K(\pi, 1)$)

$\forall\ f : M \to M$ of compact manifold, $K(\pi, 1)$-space with π virtually nilpotent $\exists\ A_{[f]} \in \mathcal{M}_{d\times d}(\mathbb{Z})$ such that

$$\mathbf{h}(f) \geq \log \mathrm{sp}(\wedge^* D_f) = \log \mathrm{sp}(\wedge^* A_{[f]}) \geq \log \mathrm{sp}(f)\,.$$

If M infranil $\Longrightarrow$ $=$ **holds for every affine** *endom. $\phi : M \to M$, a factor of an affine Φ, e.g. for ϕ_f.*
$\Longrightarrow$ $\mathbf{h}(f) \geq \mathbf{h}(\phi_f)$, ϕ_f minimizes entropy in homotopy class of f.

We define the matrices $A_{[f]}$, $D_f \in \mathcal{M}_{d\times d}(\mathbb{R})$, and ϕ_f later.
Obviously $\mathrm{sp}(\wedge^* A_{[f]}) \geq \mathrm{sp}\, A_{[f]}$ and $>$ in general.
$\exists$ an example of 6-dim. nilmanifold $N_3(\mathbb{R}) \times N_3(\mathbb{R})/\Gamma$ and a factor of autom. ϕ for which $\mathrm{sp}(\wedge^* A_{[\phi]}) > \mathrm{sp}(\phi)$ (S. Smale [1]).

Remark

The logarithm of spectral r. of exterior power $\mathrm{sp}(\wedge^* D_f)$, or $\mathrm{sp}(\wedge^* A_{[f]})$, is "a kind of volume growth" of $f_\#$, i.e an **inv**(f).

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If M infranil $\Longrightarrow$ $=$ **holds for every affine** *endom. $\phi : M \to M$, a factor of an affine Φ, e.g. for ϕ_f.*
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We define the matrices $A_{[f]}$, $D_f \in \mathcal{M}_{d\times d}(\mathbb{R})$, and ϕ_f later.
Obviously $\mathrm{sp}(\wedge^* A_{[f]}) \geq \mathrm{sp}\, A_{[f]}$ and $>$ in general.
$\exists$ an example of 6-dim. nilmanifold $N_3(\mathbb{R}) \times N_3(\mathbb{R})/\Gamma$ and a factor of autom. ϕ for which $\mathrm{sp}(\wedge^* A_{[\phi]}) > \mathrm{sp}(\phi)$ (S. Smale [1]).

Remark

The logarithm of spectral r. of exterior power $\mathrm{sp}(\wedge^* D_f)$, or $\mathrm{sp}(\wedge^* A_{[f]})$, is "a kind of volume growth" of $f_\#$, i.e an **inv**(f).

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Theorem (Not difficult - topology = de Rham invariant forms)

$\mathbf{N} = G/\Gamma$, *a nilman., con. nilp. Lie group* G *by a lattice* $\Gamma \subset G$, *and* $\phi_f : \mathbb{N} \to \mathbb{N}$ *the factor of an endom.* $\Phi_f : G \to G$, $\Phi_f(\Gamma) \subset \Gamma$.

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Proposition (Nonlinear projection - crucial)

$\exists$ a continuous map $\theta = \theta_f : G \to G^u$ which is **onto**, *moreover $\theta_{|G^u}$* **is onto** *G^u, and such that*

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$\forall\ \xi > 1\ \exists\ \ C \geq 1$ such that $\forall\ x \in G^u$, $n \in \mathbb{N}$, $x_n = \tilde{f}^n(x)$

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A construction is by shadowing ε_n-Φ-trajectory of $y_n = \tau^u\tau^{cu}(x_n)$ in G^u by a Φ-trajectory z_n. Then $\theta(x) = z$.

The proof uses technical lemmas, all of them are **nilpotent**, i.e. all errors are polynomial in the exp coordinates.

Nilpotent Alg. $\iff$ Hausdorff-Campbell formula is a polynomial.

Next we find a set S in the $B(x,r) \subset \theta(G^u)$ which is

(ξ^{2n}, n)-separated for Φ_f, $\xi > 1$, and $\#S \geq C\xi^{-2nu} \prod_{j=1}^{u} |\lambda_j|^n$.

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Maps of infra-nilmanifolds

$f : M \to M$ induces an endom. $F = F_f$ of π_M, **unique up to an inner autom.** defined by $f_\# : \pi_1(M,z) \to \pi_1(M,f(z))$.

We can consider F as **endo of** $\pi = \pi_{\mathbf{IN}}$.

Fact (K. B. Lee theorem)

$\exists$ *an* **affine self-map** $\Phi = \Phi_f = (b,B)$ *of* G, *with* $b \in G$, $B \in \mathrm{End}(G)$, *such that* $\forall\ x \in G, \alpha \in \pi_{\mathbf{IN}}$

$$\Phi(\alpha(x)) = F(\alpha)(\Phi(x)) \qquad (1)$$

By (1) $\exists$ a **factor** $\phi = \phi_f$ of **affine** Φ on **IN** by action of $\pi_{\mathbf{IN}}$ s.t.

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$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ torus	$\mathbf{N} = G/\Gamma$ nilman.	$\mathbf{IN} = G/\pi$ infranil
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Algebraic linearization

Lemma (Invariant nilpotent subgroup- [K. B. Lee, J. B. Lee])

Γ contains a subgroup $\Gamma' \lhd \pi$ such that: Γ' is nilpotent, has finite index in π, and is invariant for F.

The series of *isolators* $\sqrt[\Gamma']{\Gamma'_i} = \{x \in \Gamma' : (\exists \ell > 0)\ x^\ell \in \Gamma'_i\}$, for Γ'_i being commutators in the desc. central s. for Γ', i.e. $\Gamma'_{i+1} = [\Gamma', \Gamma'_i]$, In fact $\sqrt[\Gamma']{\Gamma'_i} = G_i \cap \Gamma'$ defined above.

Definition of Algebraic linearization

$$\mathcal{M}_{d\times d}(\mathbb{Z}) \ni A_F := \bigoplus_i A_i, \quad \text{where} \quad A_i = F : \Gamma'_{i-1}/\Gamma'_i \to \Gamma'_{i-1}/\Gamma'_i$$

Proposition

We have $A_f = A_{\tilde f}$ where $\tilde f : \tilde M \to \tilde M$ is a lift of f to the nilmanifold $\tilde M = G/\Gamma$, or from **IN** *to* **N**.

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The series of *isolators* $\sqrt[\Gamma']{\Gamma'_i} = \{x \in \Gamma' \colon (\exists \ell > 0)\ x^\ell \in \Gamma'_i\}$, for Γ'_i being commutators in the desc. central s. for Γ', i.e. $\Gamma'_{i+1} = [\Gamma', \Gamma'_i]$, In fact $\sqrt[\Gamma']{\Gamma'_i} = G_i \cap \Gamma'$ defined above.

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$$\mathcal{M}_{d\times d}(\mathbb{Z}) \ni A_F := \bigoplus_i A_i, \quad \text{where} \quad A_i = F : \Gamma'_{i-1}/\Gamma'_i \to \Gamma'_{i-1}/\Gamma'_i$$

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*We have $A_f = A_{\tilde{f}}$ where $\tilde{f} : \tilde{M} \to \tilde{M}$ is a lift of f to the nilmanifold $\tilde{M} = G/\Gamma$, or from **IN** to **N**.*

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Analytic linearization

Let $\Phi = \Phi_f = (b, B)$ be an affine self-map of G associate to $f : M \to M$ (in fact to the homom. F).
Differential $D(B)(e)$ is an **endom. of the Lie algebra** $\mathcal{G}$ of G.

Definition of Analytic linearization

Denote $D(B)(e)$ by D_f and call: **the analytical lin.** of f, i.e. F.

$\sigma(D_f) = \{\lambda_1, \dots, \lambda_d\}$ **all its eigenvalues** with multiplicities.

sp. rad. of the **full exterior power** of D_f, $\wedge^* D_f = \bigoplus_0^d \wedge^k D_f$

$$\mathrm{sp}(\wedge^* D_f) = \prod_{j:|\lambda_j|>1} |\lambda_j|, \ \lambda_j \in \sigma(D_f)$$

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provided $\exists \ |\lambda_j| > 1$.

Otherwise it is equal to 1 (in $\wedge^0 D_f$).

It is easy to show that $\sigma(A_{[f]}) = \sigma(DB(e))$, consequently:

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From general $K(\pi, 1)$ to infra-nilmanifold

Borel Conjecture: if M, M' are two manifolds being $K(\pi, 1)$ and $K(\pi', 1)$, then
$\forall$ iso. $F : \pi \to \pi'$ is induced by a homeo. $h : M \to M'$.

Proposition (Farrell-Jones theorem cf. [FaJo])

BC *holds for virt.-nilp., i.e.* M_1, M_2 *with* $\pi = \pi_1(M_1) \simeq \pi_1(M_2)$ *are homeomorphic if* π *is virtually-nilpotent.*

Remark

In fact they showed it for more general class of groups. However the dimension $d = 3$ *was not covered by the proof until the Thurston geometrization theorem [!].*

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Commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & M \end{array}$$

p a finite regular cover

Proposition (Entropy of finitely covered map)

For $(\tilde{M}, p, M)$ compact metric spaces: $\mathbf{h}(f) = \mathbf{h}(\tilde{f})$.

Proposition (Cohomology spectrum of finitely covered CW-comp.)

$\sigma(H^*(f)) \subset \sigma(H^*(\tilde{f})) \implies \mathrm{sp}(H^*(f)) \leq \mathrm{sp}(H^*(\tilde{f}))$

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Mahler measure and Lehmer conjecture

For an integer polynomial $w(x) = a_0 x^d + a_1 x^{d-1} + + \cdots + a_d$,

Definition of the Mahler measure

$$\mathbf{M}(w) := |a_0| \prod_{\lambda_i} \max(1, |\lambda_i|) \geq C \,,$$

where the product is taken over all roots of $w(x)$.

The **Lehmer's conjecture** in number theory:
there $\exists$ **a universal constant** $C > 1$, called Lehmer constant,
such that $\forall$ $w(x)$ not being a product of cyclotomic polynomials (all zeros being roots of 1) or x^k, we have

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Estimate od Mahler measure

$\exists$ estimates of the Mahler measure which depend on the **degree of an irreducible polynomial** (the **degree of an algebraic number**). An estimate given by Voutier in 1996 (cf. [Vo]),

$$\mathcal{M}(w) > \tau(d) := 1 + \frac{1}{4}\left(\frac{\log\log d}{\log d}\right)^3,$$

best known for every $d > 1$, not only asymptotically, we get:

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Smyth's theorem [Smy] (a partial answer to the Lehmer conjecture):

A polynomial of w is *non-reciprocal* $\iff$ the set of zeros is not invariant under the symmetry $\lambda \mapsto \lambda^{-1}$,

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$$\mathbf{h}(f) \geq \log \Big(\prod_{\lambda_j \in \text{roots } w_j(x)} \max(1, |\lambda_i^j|) \Big) \geq \log \tau_0 \,,$$

where τ_0 is the real root of polynomial $\tau^3 - \tau - 1$. □

The latter τ_0 is greater than **1.32471795**.
In particular, τ_0 does not depend neither on $w(x)$ nor on its deg d.

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Homotopic to Anosov diffeomorphism

Remark (Homotopy property)

Note that $\mathbf{h}(f) \geq \mathbf{h}(\phi_f)$, $f \sim \phi_f$, $A_f = A_{[f]}$
$\Longrightarrow$ *Theorem 2 is a statement about a homotopy property of* f.
A special case: A_f *is a* **hyperbolic invertible matrix over** $\mathbb{Z}$
dim. of the unstable subspace $\neq$ *dim. of the stable subspace*,
e.g. if ϕ_f *is an Anosov affine autom., and* d *dim. of* M, **is odd**.

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Corollary (Entropy of a map homotopic to an Anosov diffeomorphism)

$$\mathbf{h}(g) \geq \log 1.32471795.$$

for any map g *homotopic to an Anosov diffeomorphism of an infranil manifold* M *of* $\dim M$ *odd.*

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Homotopic to exapnsive

Definition of *forward expansive*

f is f.e. if $\exists \delta > 0$ such that $\forall x \neq y$

$$\exists n \geq 0 \text{ with } \rho(f^n(x), f^n(y)) \geq \delta$$

This implies expanding in an appropriate metric, see [PrUr]).

Remark (Expansive maps)

Finally, if f itself is metric expanding on a compact orientable manifold or at least forward expansive,
then for its degree $d(f)$ one has $\mathbf{h}(f) \geq \log |d(f)| \geq \log 2$.
Note that f expanding can happen only on infra-nilmanifolds, Franks, Shub, Gromov, Dekimpe

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