Implicit iteration processes for approximation of common fixed points of nonexpansive semigroups

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Uniform convexity and uniform smoothness

A Banach space X is called uniformly convex if or every $\varepsilon \in (0, 2]$ the modulus of convexity defined as

$$\delta(\varepsilon) = \inf\{1 - \|(x + y)/2\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$$

is strictly positive.

A Banach space X is called uniformly smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for all $x, y \in S_X$ where S_X denotes the unit sphere in X.

Typical examples of Banach spaces that are uniformly convex and uniformly smooth: l^p and $L^p[a, b]$ spaces for $1 < p < \infty$.

Inequalities in uniformly smooth Banach spaces

Let X be a uniformly smooth Banach space. Then there exists $K > 0$ such that

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \leq \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + K\|h\|^2$$

for all $x, h \in X$, where J is the normalized duality map from X to X^* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

and $\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^*.

Note: in case of smooth Banach spaces J is single-valued. In this case, we say that the duality map J is w-continuous at zero if for every $y \in X$, $\langle y, J(x_n)\rangle \to 0$ if $x_n \rightharpoonup 0$.

Strongly-continuous nonexpansive semigroup

Let X be Banach space. A one-parameter family $\mathcal{F} = \{T_t : t \in [0, \infty)\}$ of mappings from $C \subset X$ into itself is said to be a strongly-continuous nonexpansive semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in [0, \infty)$;

(iii) for each $t \in [0, \infty)$, T_t is a nonexpansive mapping;

(iv) for each $x \in C$, the mapping $t \to T_t(x)$ is strong continuous.

$F(T_t)$ denotes the set of fixed points of T_t.
The set of all common set points of the semigroup $\mathcal{F}$ is defined as

$$F(\mathcal{F}) = \bigcap_{t \in [0,\infty)} F(T_t).$$

We know that $F(\mathcal{F}) \neq \emptyset$ if X is uniformly convex, Browder 1967.

Implicit iterative processes

Given a nonexpansive semigroup $\mathcal{F} = \{T_t : t \in [0, \infty)\}$ on C, the implicit iteration process $P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\})$ is defined by

$$\begin{cases} x_0 \in C \\ x_{k+1} = c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k, \ \ for \ k \geq 0, \end{cases} \tag{1}$$

where $\{c_k\} \subset (0, 1)$ is bounded away from 0 and 1 and $\{t_k\} \subset (0, \infty)$. For fixed $k \in \mathbb{N}^0$, $w \in C$ the mapping

$$P_{k,w}(u) = c_k T_{t_{k+1}}(u) + (1 - c_k)w. \tag{2}$$

is a contraction from C into C, hence by the Banach Contraction Principle each x_{k+1} in (1) is uniquely defined.

General properties of implicit processes

Limit Existence Lemma

Let X be a uniformly convex Banach space and C be a nonempty, closed, bounded and convex subset of X. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C, $w \in F(\mathcal{F})$ and $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\})$ be an implicit iteration process. Then there exists $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.

Approximate Fixed Point Sequence (AFPS) Lemma

Under assumptions of the above lemma, we have $\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0$.

Important technique in uniformly convex Banach spaces

Lemma

Let X be a uniformly convex Banach space. Let $\{c_n\} \subset (0,1)$ be bounded away from 0 and 1, and $\{u_n\}, \{v_n\} \subset X$ be such that

$$\limsup_{n\to\infty} \|u_n\| \le a, \ \limsup_{n\to\infty} \|v_n\| \le a, \ \lim_{n\to\infty} \|c_n u_n + (1-c_n)v_n\| = a.$$

Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Normalized implicit iterative processes

Definition

Normalized IIP is defined as $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$, where $P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\})$ is an implicit iterative process, $\{k_i\}$ is a strictly increasing sequence of natural numbers such that $\lim_{i\to\infty} t_{k_i} = 0$, $\lim_{i\to\infty} \frac{1}{t_{k_i}} \|T_{t_{k_i}}(x_{k_i}) - x_{k_i}\| = 0$, and $w_i = x_{k_i}$.

There are known examples of such normalizations, e.g. take $\{t_n\}$ such that: $t_n > 0$ for every $n \in \mathbb{N}$, $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Fix $0 < \eta < \limsup_{n\to\infty} t_n$. Using a technical density lemma for real numbers and AFPS Lemma, there exists $\{k_i\} \subset \mathbb{N}$ such that $\frac{\eta}{2^{i+1}} < t_{k_i} < \frac{\eta}{2^{i-1}}$ and $\|T_{t_{k_i}}(x_{k_i}) - x_{k_i}\| < \frac{\eta}{2^{2(i+1)}}$.

Our goal result

Weak Convergence Theorem

Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X with a w-continuous at zero duality map. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$ be a normalized implicit iterative process. Then there exists a common fixed point $w \in F(\mathcal{F})$ such that $w_i \rightharpoonup w$.

Our goal result - Steps of proof

Weak Convergence Theorem

Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X with a w-continuous at zero duality map. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$ be a normalized implicit iterative process. Then there exists a common fixed point $w \in F(\mathcal{F})$ such that $w_i \rightharpoonup w$.

Step 1: Show that every weak cluster point of $\{w_i\}$ belongs to $F(\mathcal{F})$.
Step 2: Consider two subsequences $\{w_{\alpha_n}\}$ and $\{w_{\beta_n}\}$ of $\{w_i\}$ such that $w_{\alpha_n} \rightharpoonup y$, $w_{\beta_n} \rightharpoonup z$. By Step 1, $y \in F(\mathcal{F})$ and $z \in F(\mathcal{F})$.
Step 3: Show that $y = z$. Therefore, $\{w_i\}$ has at most one weak cluster point.
Step 4: Since C is weakly compact, $\{w_i\}$ has a weak cluster in C, hence by Step 3, $w_i \rightharpoonup w$. By Step 1, $w \in F(\mathcal{F})$. □

Demiclosedness Principle

"Demiclosedness Principle" Theorem

Let X be a uniformly convex and uniformly smooth Banach space with a w-continuous at zero duality map. Let C be a nonempty, closed, bounded and convex subset of X. Assume $\{s_i\}_{i\in\mathbb{N}}$ to be a sequence of real numbers such that $s_i > 0$ for all $i \in \mathbb{N}$ and $s_i \to 0$. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C, $\{w_i\}_{i\in\mathbb{N}} \subset C$, $w \in C$ be such that $w_i \rightharpoonup w$ and $\lim\limits_{i\to\infty} \dfrac{1}{s_i}\|T_{s_i}(w_i) - w_i\| = 0$. Then $w \in F(\mathcal{F})$.

Corollary: If process $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$ is normalized then every weak cluster point of $\{w_i\}$ belongs to $F(\mathcal{F})$.

Demiclosedness Principle - Proof Steps

"Demiclosedness Principle" Theorem

Let X be a uniformly convex and uniformly smooth Banach space with a w-continuous at zero duality map. Let C be a nonempty, closed, bounded and convex subset of X. Assume $\{s_i\}_{i\in\mathbb{N}}$ to be a sequence of real numbers such that $s_i > 0$ for all $i \in \mathbb{N}$ and $s_i \to 0$. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C, $\{w_i\}_{i\in\mathbb{N}} \subset C$, $w \in C$ be such that $w_i \rightharpoonup w$ and $\lim_{i\to\infty} \frac{1}{s_i}\|T_{s_i}(w_i) - w_i\| = 0$. Then $w \in F(\mathcal{F})$.

Step 1: Define $\varphi(u) = \limsup_{i\to\infty} \|w_i - u\|$. Show $\varphi(w) = \inf_{x\in C} \varphi(x)$.
Step 2: Fix $t > 0$. Show that $\varphi(w) = \varphi(T_t(w))$.
Step 3: Use parallelogram inequality to prove
$\|w_i - \frac{1}{2}(w + T_t(w))\|^2 \leq \frac{1}{2}\|w_i - w\|^2 + \frac{1}{2}\|w_i - T_t(w)\|^2 - \frac{1}{4}\lambda(\|T_t(w) - w\|)$.
Step 4: Use Steps 1,2,3 to get
$0 \leq \lambda(\|T_t(w) - w\|) \leq 2\varphi(T_t(w))^2 - 2\varphi(w)^2 = 0$. Hence $T_t(w) = w$. $\square$

Parallelogram inequality

Let X be a uniformly convex Banach space X. For each $d > 0$ there exists a continuous function $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(t) = 0$ if and only if $t = 0$, and

$$\|cx + (1-c)y\|^2 \leq \; c\|x\|^2 + (1-c)\|y\|^2 - c(1-c)\lambda(\|x-y\|),$$

for any $c \in [0, 1]$ and all $x, y \in X$ such that $\|x\| \leq d$ and $\|x\| \leq d$.

Note: This inequality is a generalization of the Parallelogram Law in Hilbert spaces

$$\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2.$$

Our goal result - Steps 1 and 2

Weak Convergence Theorem

Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X with a w-continuous at zero duality map. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$ be a normalized implicit iterative process. Then there exists a common fixed point $w \in F(\mathcal{F})$ such that $w_i \rightharpoonup w$.

Step 1: Show that every weak cluster point of $\{w_i\}$ belongs to $F(\mathcal{F})$.
Step 2: Consider two subsequences $\{w_{\alpha_n}\}$ and $\{w_{\beta_n}\}$ of $\{w_i\}$ such that $w_{\alpha_n} \rightharpoonup y$, $w_{\beta_n} \rightharpoonup z$. By Step 1, $y \in F(\mathcal{F})$ and $z \in F(\mathcal{F})$.

Step 3: Show that $y = z$. Therefore, $\{w_i\}$ has at most one weak cluster point.

Step 4: Since C is weakly compact, $\{w_i\}$ has a weak cluster in C, hence by Step 3, $w_i \rightharpoonup w$. By Step 1, $w \in F(\mathcal{F})$.

Lemma on Existence of Limit

Lemma on Existence of Limit

Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\})$ be an implicit iterative process. Let $w_1,\ w_2 \in F(\mathcal{F})$, $0 \leq t \leq 1$. Then there exists $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|tx_k + (1-t)w_1 - w_2\| = r$.

Bruck Lemma

Let X be a uniformly convex Banach space, and let $C \subset X$ be nonempty, bounded, closed and convex. There exists a strictly increasing, convex continuous function $\gamma_2 : [0,\infty) \to [0,\infty)$ with $\gamma_2(0) = 0$ such that for every nonexpansive $T : C \to C$, $c \in [0,1]$ and every $x, y \in C$ we have

$$\gamma_2\Big(\|T(cx + (1-c)y) - cT(x) - (1-c)T(y)\|\Big) \leq \|x - y\| - \|T(x) - T(y)\|.$$

Weak Cluster Point Lemma

Weak Cluster Point Lemma

Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\})$ be an implicit iterative process. Let $w_1,\ w_2 \in F(\mathcal{F})$ and $y,\ z$ be two weak cluster points of $\{x_k\}_{k\in\mathbb{N}^0}$. Then $\langle y - z, J(w_1 - w_2)\rangle = 0$.

Note: if $y,\ z$ be two weak cluster points of $\{x_k\}_{k\in\mathbb{N}^0}$ and $y,\ z \in F(\mathcal{F})$ then from Lemma we have $0 = \langle y - z, J(y - z)\rangle = \|y - z\|^2$.

Our goal result - Final

Weak Convergence Theorem

Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X with a w-continuous at zero duality map. Let $\mathcal{F}$ be a strongly-continuous nonexpansive semigroup on C and $\{w_i\}_{k\in\mathbb{N}^0} = P(C, \mathcal{F}, x_0, \{c_k\}, \{t_k\}, \{k_i\})$ be a normalized implicit iterative process. Then there exists a common fixed point $w \in F(\mathcal{F})$ such that $w_i \rightharpoonup w$.

Step 1: Show that every weak cluster point of $\{w_i\}$ belongs to $F(\mathcal{F})$.
Step 2: Consider two subsequences $\{w_{\alpha_n}\}$ and $\{w_{\beta_n}\}$ of $\{w_i\}$ such that $w_{\alpha_n} \rightharpoonup y$, $w_{\beta_n} \rightharpoonup z$. By Step 1, $y \in F(\mathcal{F})$ and $z \in F(\mathcal{F})$.
Step 3: Show that $y = z$. Therefore, $\{w_i\}$ has at most one weak cluster point.
Step 4: Since C is weakly compact, $\{w_i\}$ has a weak cluster in C, hence $w_i \rightharpoonup w$. By Step 1, $w \in F(\mathcal{F})$.

Further Research Opportunities

Let X be a Banach space, $C \subset X$ bounded closed convex

Pointwise Lipschitzian Semigroup

A semigroup $\mathcal{F} = \{T_t : t \in J\}$ of mappings from C into itself is said to be a pointwise Lipschitzian semigroup on C if for each $t \in J$, T_t is a pointwise Lipschitzian mapping, i.e. there exists a function $\alpha_t : C \to [0, \infty)$ such that

$$\|T_t(x) - T_t(y)\| \leq \alpha_t(x)\|x - y\| \ for\ all\ x, y \in C.$$

$\mathcal{F}$ is said to be asymptotically nonexpansive if $\limsup_{t\to\infty} \alpha_t(x) \leq 1$ for every $x \in C$.

Proposition: Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup on C. Assume that all mappings $T_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C then $\mathcal{F}$ is asymptotically nonexpansive on C if and only if for each $x \in C$

$$\limsup_{t\to\infty} \|(T_t)_x'\| \leq 1.$$

Pointwise Lipschitzian Mappings

Pointwise Lipschitzian Mappings - Definition

Let $C \neq \emptyset$ be a subset of a Banach space X. $T : C \to C$ is called pointwise Lipschitzian if there exists a mapping $\alpha : C \to [0, 1]$ such that $\|T(x) - T(y)\| \leq \alpha(x)\|x - y\|$ for all $x, y \in C$.

This look 'asymmetrical' at first glance but we all know an example from the calculus class 'Intermediate value theorem'. Let $C = [c, d]$ and $x, y \in [c, d]$, $y < x$. Then there exists $y \leq z \leq x$ such that

$$\frac{T(x) - T(y)}{x - y} = T'(z).$$

Hence, $|T(x) - T(y)| \leq \max\limits_{w \in [c,x]} |T'(w)| \; |x - y|$, so $a(x) = \max\limits_{w \in [c,x]} |T'(w)|$.
At the same time, $|T(x) - T(y)| \leq \max\limits_{w \in [y,d]} |T'(w)| \; |x - y|$ which give us the pointwise Lipschitz coefficient w.r.t. y.

Common Fixed Points for Asymptotically Nonexpansive Pointwise Lipschitzian Semigroup

Existence Theorem

Assume X is uniformly convex. Let $\mathcal{F}$ be an asymptotically nonexpansive pointwise Lipschitzian semigroup on C. Then the set $F(\mathcal{F})$ of common fixed points is nonempty, closed, and convex.

W.M. Kozlowski, Common fixed points for semigroups of pointwise Lipschitzian mappings in Banach spaces, Bull. Austral. Math Soc., 84 (2011), 353 - 361.

Common Fixed Points for Asymptotically Nonexpansive Pointwise Lipschitzian Semigroup

Construction Results - Mann and Ishikawa Iterative Processes

Case of X uniformly convex and uniformly smooth:
W.M. Kozlowski, On the construction of common fixed points for semigroups of nonlinear mappings in uniformly convex and uniformly smooth Banach spaces, Comment. Math., 52.2 (2012), 113-136

Case of X uniformly convex with Opial property:
W.M. Kozlowski, B. Sims, On the convergence of iteration processes for semigroups of nonlinear mappings in Banach spaces, Computational and Analytical Mathematics. In Honor of Jonathan Borweins 60th Birthday. Springer Proc. Math. Stat. 50 (2013),463-484

Modular Function Spaces - MFS

Modular Function Spaces - generalization of both function and sequence variants of many spaces, like Lebesgue, Köthe, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz. The importance for applications - the richness of structure of MFS, that - besides being Banach spaces - are equipped with modular equivalents of norm or metric notions. They are also equipped with almost everywhere convergence and convergence in submeasure, and with the natural ordering.

W.M. Kozlowski, "Modular Function Spaces", M. Dekker 1988

W.M. Kozlowski, "An Introduction to Fixed Point Theory in Modular Function Spaces", in "Topics in Fixed Point Theory", Springer 2014

M.A. Khamsi and W.M. Kozlowski, "Fixed Point Theory in Modular Function Spaces", Springer Birkhäuser, to be launched at the AMS Joint Mathematics Meeting, San Antonio, Texas (USA), 10 - 13 January 2015

Thank You

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Thank You for Your Attention!
Questions?